

# Overcomplete Tensor Decomposition via Convex Optimization

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## Abstract

Tensors provide natural representations for massive multi-mode datasets and tensor methods also form the backbone of many machine learning, signal processing, and statistical algorithms. The utility of tensors is mainly due to the ability to identify *overcomplete, non-orthogonal* factors from tensor data, which is known as tensor decomposition. This work develops theories and computational methods for guaranteed *overcomplete, non-orthogonal* tensor decomposition using convex optimization. We consider tensor decomposition as a problem of measure estimation from moments. We develop the theory for guaranteed decomposition under three assumptions: (i) Incoherence; (ii) Bounded spectral norm; and (iii) Gram isometry. We show that under these three assumptions, one can retrieve tensor decomposition by solving a convex, infinite-dimensional analog of  $\ell_1$  minimization on the space of measures. The optimal value of this optimization defines the tensor nuclear norm that can be used to regularize tensor inverse problems, including tensor completion, denoising, and robust tensor principal component analysis. Remarkably, all the three assumptions are satisfied with high probability if the rank-one tensor factors are uniformly distributed on the unit spheres, implying exact decomposition for tensors with random factors. We also present and numerically test two computational methods based respectively on Burer-Monteiro low-rank factorization reformulation and the Sum-of-Squares relaxations.

## 1 Introduction

Tensors provide natural representations for massive multi-mode datasets encountered in image and video processing [1], collaborative filtering [2], array signal processing [3], and psychometrics [4]. Tensor methods also form the backbone of many machine learning, signal processing, and statistical algorithms, including independent component analysis (ICA) [5, 6], latent graphical model learning [7], dictionary learning [8], and Gaussian mixture estimation [9]. The utility of tensors in such diverse applications is mainly due to the ability to identify *overcomplete, non-orthogonal* factors from tensor data as already suggested by Kruskal's theorem [10]. This is known as tensor decomposition, which describes the problem of decomposing a tensor into a linear combination of a small number of rank one tensors. This is in sharp contrast to the inherent ambiguous nature of matrix decompositions without additional assumptions such as orthogonality and nonnegativity.

In addition to its practical applicability, tensor decomposition is also of fundamental theoretical interest in solving linear inverse problems involving low-rank tensors. For one thing, theoretical results for tensor decomposition inform what types of rank-one tensor combinations are identifiable given full observations. For another, the dual polynomial is constructed to certify a particular decomposition, which is useful in investigating the regularization power of tensor nuclear norm for tensor inverse problems. These problems

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include completion, denoising, and robust tensor principal component analysis. Certifying dual polynomial constructed in this work will play a role in tensor completion similar to the role played by the subdifferential characterization of matrix nuclear norm in matrix completion and low-rank matrix recovery [11, 12].

Despite the similarity between matrix and tensor problems, the latter are significantly more challenging due to two things. First, the inherent difficulty associated with tensor problems – in fact, most tensor problems are NP hard [13]. Second, it is due to the lack of proper theories for basic tensor concepts and operations such as singular values, vectors, and singular value decompositions. To address these challenges, we employ the perspective of measure estimation from moments to solve tensor decomposition. Tensor decomposition aims to decompose a given tensor into a linear combination of a small number of rank-one tensors. The decomposition with the smallest rank is called a Canonical Polyadic (CP) decomposition and the corresponding number of rank-one tensors is the rank of the tensor. Consider the following decomposition of a third order non-symmetric tensor

$$T = \sum_{p=1}^r \lambda_p^* \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* \in \mathbb{R}^{n_1 \times n_2 \times n_3}. \quad (1.1)$$

Here the factors  $\{(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)\}_{p=1}^r$  are *overcomplete* (that is, the rank  $r$  is greater than the individual tensor dimensions  $n_1, n_2$  and  $n_3$ ), *non-orthogonal* and live on the unit spheres. Without loss of generality, we also assume that the coefficients  $\lambda_p^*$ 's are positive as their signs can always be absorbed into the factors. Retrieving the decomposition from the observed tensor entries in  $T$  is equivalent to recovering an atomic measure

$$\mu^* = \sum_{p=1}^r \lambda_p^* \delta(\mathbf{u} - \mathbf{u}_p^*, \mathbf{v} - \mathbf{v}_p^*, \mathbf{w} - \mathbf{w}_p^*) \quad (1.2)$$

supported on the product of unit spheres  $\mathbb{K} := \mathbb{S}^{n_1-1} \times \mathbb{S}^{n_2-1} \times \mathbb{S}^{n_3-1}$  from its third order moments

$$T = \int_{\mathbb{K}} \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} d\mu^*. \quad (1.3)$$

In most practical scenarios, we are interested in a low-rank tensor where  $r$  is much smaller than the product  $n_1 n_2 n_3$  (but can be significantly larger than individual  $n_1, n_2$ , and  $n_3$ ). Therefore, the decomposition (1.1) is *sparse*.

There are several advantages offered by this point of view. First, it provides a natural way to extend the  $\ell_1$  minimization in finding sparse representations for finite dictionaries [14] to tensor decomposition. By viewing the set of rank-one tensors  $\mathcal{A} = \{\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} : (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{K}\}$  as a dictionary with an infinite number of atoms, this formulation finds the sparsest representation of  $T$  by minimizing the  $\ell_1$  norm with respect to the infinite dictionary  $\mathcal{A}$ . More precisely, we recover  $\mu^*$  from the tensor  $T$  by solving the following optimization

$$\underset{\mu \in \mathcal{M}(\mathbb{K})}{\text{minimize}} \mu(\mathbb{K}) \text{ subject to } T = \int_{\mathbb{K}} \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} d\mu, \quad (1.4)$$

where  $\mathcal{M}(\mathbb{K})$  is the set of (non-negative) Borel measures on  $\mathbb{K}$ , and  $\mu(\mathbb{K})$  is the total measure/mass of the set  $\mathbb{K}$ . Second, the optimal value of the total mass minimization defines precisely the *tensor nuclear norm*, which is a special case of atomic norms [15] corresponding to the atomic set  $\mathcal{A}$ . The tensor nuclear norm is useful in many tensor inverse problems like tensor completion, robust tensor principal component analysis, and stable tensor recovery.

The major theoretical problem we investigate is under what conditions on the rank-one factors  $\{(\mathbf{u}_k^*, \mathbf{v}_k^*, \mathbf{w}_k^*)\}$ , the total mass minimization returns the decomposition (1.1). These conditions form the major part of the main result of this work. For ease of presentation, we assume a square tensor with  $n_1 = n_2 = n_3 = n$ . Before presenting the main theorem, we list the three assumptions we are building on.

**Assumption I: Incoherence.** The tensor factors are incoherent, *i.e.*, the incoherence  $\Delta$  defined below satisfies

$$\Delta := \max_{p \neq q} \{|\langle \mathbf{u}_p^*, \mathbf{u}_q^* \rangle|, |\langle \mathbf{v}_p^*, \mathbf{v}_q^* \rangle|, |\langle \mathbf{w}_p^*, \mathbf{w}_q^* \rangle|\} \leq \frac{\tau(\log n)}{\sqrt{n}} \quad (1.5)$$

where  $\tau(\cdot)$  is a polynomial function of its argument.

**Assumption II: Bounded spectral norm.** The spectral norms of  $U := [\mathbf{u}_1^* \ \cdots \ \mathbf{u}_r^*]$ ,  $V := [\mathbf{v}_1^* \ \cdots \ \mathbf{v}_r^*]$ ,  $W := [\mathbf{w}_1^* \ \cdots \ \mathbf{w}_r^*]$  are well-controlled

$$\max\{\|U\|, \|V\|, \|W\|\} \leq 1 + c\sqrt{\frac{r}{n}} \quad (1.6)$$

for some numerical constant  $c > 0$ .

**Assumption III: Gram isometry.** The Hadamard product (denoted as  $\odot$ ) of the Gram matrices of  $U$  and  $V$  satisfies an isometry condition:

$$\|(U^\top U) \odot (V^\top V) - I_r\| \leq \kappa(\log n) \frac{\sqrt{r}}{n} \quad (1.7)$$

where  $\kappa(\cdot)$  is a polynomial. Similar bounds hold for  $U, W$ , and  $V, W$  (without loss of generality with the same polynomial  $\kappa(\cdot)$ ).

Now we are ready to present the main result of this work in the following theorem:

**Theorem 1.1.** Suppose the tensor  $T \in \mathbb{R}^{n_1 \times n_2 \times n_3}$  admits a decomposition given in (1.1). Under Assumptions I, II, III and in addition, if

$$r \leq \frac{n^{17/16}}{12c^2 \sqrt{\tau(\log n)}},$$

then for sufficiently large  $n$ ,  $\mu^*$  given in (1.2) is the unique optimal solution of (1.4).

A few remarks follow. Firstly, since  $r = O(n^{17/16}/\sqrt{\tau(\log n)}) \gg n$ , total mass minimization is guaranteed to recover *overcomplete* tensor decompositions. Secondly, the incoherence condition is reasonable as we argue in the following. Tensor decomposition using total mass minimization is an atomic decomposition problem. The latter determines the conditions under which a decomposition in terms of atoms in an atomic set  $\mathcal{A}$  achieves the corresponding atomic norm. For example, the singular value decomposition is an atomic decomposition for the set of unit-norm, rank-one matrices. As shown in [16], for a large class of atomic sets, only decompositions composed of sufficiently *different* atoms are valid atomic decompositions. In particular, a necessary condition for tensor atomic decomposition is that the incoherence  $\Delta \leq \cos(\frac{2}{3})$ . However, our sufficient incoherence condition (1.5) is still significantly stronger than the necessary condition. Finally, if the incoherence bound in Assumption I is further strengthened to  $O(\frac{1}{n\alpha(\log n)})$  for some polynomial  $\alpha(\cdot)$ , then Assumptions II and III are consequences of Assumption I. So if the rank-one factors of an overcomplete tensor are incoherent enough, without needing Assumptions II and III, its CP decomposition can always be uniquely identified.

It is worth commenting on the relationship between our theorem and the classical Kruskal's uniqueness theorem for tensor decompositions. The Kruskal rank of a size- $n \times r$  matrix  $U$  is defined as the maximal number  $k_U$  such that any  $k_U$  out of  $U$ 's  $r$  columns are linearly independent. Kruskal's theorem states that if  $r$  in the expansion (1.1) satisfies

$$r \leq \frac{1}{2}(k_U + k_V + k_W) - 1$$

then  $T$  has CP rank  $r$  and its expression as a rank  $r$  tensor is unique (up to permutation and sign ambiguities). Since  $k_U, k_V, k_W \leq n$  for matrices in  $\mathbb{R}^{n \times r}$  and the inequalities are achieved for generic matrices  $U, V$  and  $W$ , Kruskal's theorem ensures unique decomposition involving up to  $r = \frac{3}{2}n - 1$  rank-one (generic) factors. Note that our result holds for  $r$  up to the order  $n^{17/16}$ , which can be significantly larger than  $\frac{3}{2}n$  for large  $n$ . In the regime  $r \leq \frac{3}{2}n - 1$ , however, one might wonder whether Theorem 1.1 is trivial given the uniqueness of the decomposition. This is not the case. The caveat here is that the uniqueness holds when restricted to decompositions involving exactly  $r$  terms, while the tensor nuclear norm, *i.e.*, the optimal value of (1.4),

can potentially be achieved by decompositions involving more than  $r$ , even an infinite number of terms. In fact, the formulation takes into account decompositions with continuous supports. Theorem 1.1 excludes such possibility under the given conditions.

Finally, we remark that Assumptions I, II and III are satisfied with high probability if the rank-one factors  $\{\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*\}_{p=1}^r$  are generated independently according to uniform distributions on the unit spheres [17], leading to the following corollary:

**Corollary 1.1.** *If the rank-one factors  $\{(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)\}_{p=1}^r$  are generated independently according to uniform distributions on the unit spheres, and if  $r \leq \frac{n^{17/16}}{12c^2\sqrt{\tau(\log n)}}$ , then for sufficiently large  $n$ , solving optimization (1.4) is guaranteed to recover  $\mu^*$  with high probability.*

Corollary 1.1 is also justified by numerical experiments in Section 6. In the experiments, we randomly sampled the unit spheres to generate the true factors of the tensor and then applied our proposed approach to decompose it. We will see that in this case one can exactly recover the factors even for  $r \gg n$ .

The rest of the paper is organized as follows. In Section 2, we connect tensor decomposition to atomic decomposition, apply duality theory to derive a sufficient condition for exact decomposition, and describe extensions of the framework to tensor inverse problems. Section 3 presents computational methods to solve the tensor decomposition. We describe connections to prior art and the foundations that we build upon in Section 4. We then proceed to develop proofs in Section 5. In Section 6, we validate our theory using numerical experiments.

## 2 Tensor Decomposition, Atomic Norms, and Duality

### 2.1 Tensor Decomposition as an Atomic Decomposition

In this work, we view tensor decomposition in the frameworks of both atomic norms and measure estimation. The unit sphere of  $\mathbb{R}^n$  is denoted by  $\mathbb{S}^{n-1}$ , and the direct product of three unit spheres  $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$  by  $\mathbb{K}$ . The tensor atomic set is denoted by  $\mathcal{A} = \{\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} : (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{K}\}$  parameterized by the set  $\mathbb{K}$ , where  $\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}$  is a rank-one tensor with the  $(i, j, k)$ th entry being  $u_i v_j w_k$ . For any tensor  $T$ , its atomic norm with respect to  $\mathcal{A}$  is defined by [15]:

$$\begin{aligned} \|T\|_{\mathcal{A}} &= \inf\{t : T \in t \operatorname{conv}(\mathcal{A})\} \\ &= \inf \left\{ \sum_p \lambda_p : T = \sum_{p=1}^r \lambda_p \mathbf{u}_p \otimes \mathbf{v}_p \otimes \mathbf{w}_p, \lambda_p > 0, (\mathbf{u}_p, \mathbf{v}_p, \mathbf{w}_p) \in \mathbb{K} \right\}, \end{aligned} \quad (2.1)$$

where  $\operatorname{conv}(\mathcal{A})$  is the convex hull of the atomic set  $\mathcal{A}$ , and a scalar multiplying a set scales every element in the set. Therefore, the tensor atomic norm is the minimal  $\ell_1$  norm of the expansion coefficients among all valid expansions in terms of unit-norm, rank-one tensors. The atomic norm  $\|T\|_{\mathcal{A}}$  defined in (2.1) is also called the tensor nuclear norm and denoted by  $\|T\|_*$ . We will use these two names and notations interchangeably in the following. The way we define the tensor nuclear norm is precisely the way the matrix nuclear norm is defined.

We argue that the two lines in the definition (2.1) are consistent and are also equivalent to (1.4). Since  $\operatorname{conv}(\mathcal{A}) = \{T : T = \int_{\mathbb{K}} \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} d\mu, \mu \in \mathcal{M}(\mathbb{K}), \mu(\mathbb{K}) \leq 1\}$ , the first line in the definition (2.1) implies that  $\|T\|_{\mathcal{A}}$  is equal to the optimal value of (1.4). Compared with the measure optimization (1.4), the feasible region of the minimization defining the atomic norm in the second line is restricted to discrete measures. However, these two optimizations share the same optimal value as a consequence of the Carathéodory's convex hull theorem, which states that if a point  $\mathbf{x} \in \mathbb{R}^d$  lies in the convex hull of a set, then  $\mathbf{x}$  can be written as a convex combination of at most  $d+1$  points of that set [18]. Since  $T \in \|T\|_{\mathcal{A}} \operatorname{conv}(\mathcal{A}) = \operatorname{conv}(\|T\|_{\mathcal{A}} \mathcal{A})$ ,  $T$  can be expressed as a convex combination of at most  $n^3 + 1$  points of the set  $\|T\|_{\mathcal{A}} \mathcal{A}$ , implying that the optimal value is achieved by a discrete measure with support size at most  $n^3 + 1$ . This argument establishes that

the two lines in (2.1) as well as the measure optimization (1.4) are equivalent. Therefore, the atomic norm framework and the measure optimization framework are two different formulations of the same problem, with the former setting the stage in the finite dimensional space and the latter in the infinite-dimensional space of measures.

Given an abstract atomic set, the problem of atomic decomposition seeks the conditions under which a decomposition in terms of the given atoms achieves the atomic norm. In this sense, the tensor decomposition considered in this work is an atomic decomposition problem.

## 2.2 Duality

Duality plays an important role in analyzing atomic tensor decomposition. We again approach duality from both perspectives of atomic norms and measure estimation.

First of all, we find the dual problem of optimization (1.4). Given  $Q, T \in \mathbb{R}^{n \times n \times n}$ , we define the tensor inner product  $\langle Q, T \rangle := \sum_{i,j,k} Q_{ijk} T_{ijk}$ . Standard Lagrangian analysis shows that the dual problem of (1.4) is the following semi-infinite problem, which has an infinite number of constraints:

$$\begin{aligned} & \underset{Q \in \mathbb{R}^{n \times n \times n}}{\text{maximize}} \quad \langle Q, T \rangle \\ & \text{subject to} \quad \langle Q, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \rangle \leq 1, \forall (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{K}. \end{aligned} \quad (2.2)$$

The polynomial  $q(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \langle Q, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \rangle = \sum_{i,j,k} Q_{ijk} u_i v_j w_k$  corresponding to a dual feasible solution is called a dual polynomial. The dual polynomial associated with an optimal dual solution can be used to certify the optimality of a particular decomposition, as demonstrated by the following proposition.

**Proposition 2.1.** *Let the support of the true atomic measure  $\mu^*$  be  $S := \{(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)\}_{p=1}^r$ . Suppose the set of rank-one tensor factors  $\{\mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^*\}_{p=1}^r$  are linearly independent. If there exists a dual solution  $Q \in \mathbb{R}^{n \times n \times n}$  such that the corresponding dual polynomial*

$$q(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \langle Q, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \rangle \quad (2.3)$$

*satisfies the following Bounded Interpolation Property:*

$$\begin{aligned} q(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*) &= 1, p = 1, \dots, r \quad (\text{Interpolation}) \\ q(\mathbf{u}, \mathbf{v}, \mathbf{w}) &< 1, \text{ for } (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{K}/S \quad (\text{Boundedness}) \end{aligned}$$

*then the decomposition in (1.1) is the unique optimal solution to (1.4).*

*Proof.* Any  $Q$  that satisfies the conditions of the proposition is a feasible solution to (2.2). We also have

$$\begin{aligned} \langle Q, T \rangle &= \left\langle Q, \sum_{p=1}^r \lambda_p^* \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* \right\rangle \\ &= \sum_{p=1}^r \lambda_p^* \langle Q, \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* \rangle \\ &= \sum_{p=1}^r \lambda_p^* q(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*) \\ &= \mu^*(\mathbb{K}), \end{aligned}$$

establishing a zero-duality gap of the primal-dual feasible solution  $(\mu^*, Q)$ . As a consequence,  $\mu^*$  is a primal optimal solution and  $Q$  is a dual optimal solution.

For uniqueness, suppose  $\hat{\mu}$  is another primal optimal solution. If  $\hat{\mu}(\mathbb{K}/S) > 0$ , then

$$\begin{aligned}\mu^*(\mathbb{K}) &= \langle Q, T \rangle = \left\langle Q, \int_{\mathbb{K}} \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} d\hat{\mu} \right\rangle \\ &= \sum_{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in S} \hat{\mu}(\mathbf{u}, \mathbf{v}, \mathbf{w}) q(\mathbf{u}, \mathbf{v}, \mathbf{w}) + \int_{\mathbb{K}/S} q(\mathbf{u}, \mathbf{v}, \mathbf{w}) d\hat{\mu} \\ &< \sum_{(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*) \in S} \hat{\lambda}_p + \int_{\mathbb{K}/S} 1 d\hat{\mu} \\ &= \hat{\mu}(\mathbb{K})\end{aligned}$$

due to the condition  $q(\mathbf{u}, \mathbf{v}, \mathbf{w}) < 1$  for  $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \notin S$ , contradicting with the optimality of  $\hat{\mu}$ . So all optimal solutions are supported on  $S$ . If  $\{\mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^*\}_{p=1}^r$  are linearly independent, the coefficients  $\{\lambda_p\}_{p=1}^r$  are also uniquely determined from  $T = \sum_{p=1}^r \lambda_p \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^*$ .  $\square$

For obvious reasons, the dual optimal solution  $Q$  is called a dual certificate. A dual certificate, or the corresponding dual polynomial satisfying the Bounded Interpolation Property (BIP), is used frequently as the starting point to derive several atomic decomposition and super-resolution results [19, 20, 21, 22]. In Section 5, we will explicitly construct a dual certificate to prove Theorem 1.1.

We next reinterpret duality and Proposition 2.1 in the atomic norm framework. The dual norm of the tensor nuclear norm, i.e., the tensor spectral norm, of a tensor  $Q$  is given by

$$\|Q\| := \sup_{T: \|T\|_* \leq 1} \langle Q, T \rangle = \sup_{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{K}} \langle Q, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \rangle.$$

The equality is due to the fact that the atomic set  $\mathcal{A}$  are the extremal points of the unit nuclear norm ball  $\{T : \|T\|_* \leq 1\}$ . In light of the spectral norm definition, we rewrite the dual problem (2.2) as

$$\underset{Q \in \mathbb{R}^{n \times n \times n}}{\text{maximize}} \langle Q, T \rangle \text{ subject to } \|Q\| \leq 1, \quad (2.4)$$

which is precisely the definition of the dual norm of the tensor spectral norm, i.e., the tensor nuclear norm.

The subdifferential (the set of subgradient) of the tensor nuclear norm is defined by

$$\partial\|T\|_* = \{Q \in \mathbb{R}^{n \times n \times n} : \|R\|_* \geq \|T\|_* + \langle R - T, Q \rangle, \text{ all } R \in \mathbb{R}^{n \times n \times n}\}. \quad (2.5)$$

The subdifferential has an equivalent representation [23]:

$$\partial\|T\|_* = \{Q \in \mathbb{R}^{n \times n \times n} : \|T\|_* = \langle Q, T \rangle, \|Q\| \leq 1\}. \quad (2.6)$$

For  $T$  having an atomic decomposition given in (1.1), it can be established that the defining properties of (2.6) are equivalent to

$$\begin{aligned}\langle Q, \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* \rangle &= 1, \quad p = 1, \dots, r, \\ \langle Q, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \rangle &\leq 1, \quad \text{for } (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{K}.\end{aligned}$$

We recognize that the BIP conditions are strengthened versions of the above conditions. Therefore, any  $Q$  satisfying the BIP is an element of the subdifferential  $\partial\|T\|_*$ . The BIP in fact means that  $Q$  is an interior point of  $\partial\|T\|_*$ . Our proof strategy for Theorem 1.1 is to construct such an interior point in Section 5. This is in contrast to the matrix case, for which we have an explicit characterization of the entire subdifferential of the nuclear norm using the singular value decomposition (more explicit than the one given in (2.6)) [12]. More specifically, suppose  $X = U\Sigma V^\top$  is the (compact) singular value decomposition of  $X \in \mathbb{R}^{m \times n}$  with  $U \in \mathbb{R}^{m \times r}$ ,  $V \in \mathbb{R}^{n \times r}$  and  $\Sigma$  being an  $r \times r$  diagonal matrix. Then the subdifferential of the matrix nuclear norm at  $X$  is given by

$$\partial\|X\|_* = \{UV^\top + W : W \text{ and } X \text{ have orthogonal row and column spaces, and } \|W\| \leq 1\}.$$

It is challenging to obtain such a characterization for tensors unless the tensor admits an orthogonal rank-one decomposition.

### 2.3 Extension: Regularization Using Tensor Nuclear Norm

Independent from practical considerations, we investigate tensor decomposition for theoretical reasons. Similar to regularizing matrix inverse problems using the matrix nuclear norm, the tensor nuclear norm can be used to regularize tensor inverse problems. Suppose we observe an unknown low-rank tensor  $T^*$  through the linear measurement model  $\mathbf{y} = \mathcal{B}(T^*)$ , we would like to recover the tensor  $T^*$  from the observation  $\mathbf{y}$ . For instance, when  $\mathcal{B}$  samples the individual entries of  $T^*$ , we are looking at a tensor completion problem. We propose recovering  $T^*$  by solving

$$\underset{T \in \mathbb{R}^{n \times n \times n}}{\text{minimize}} \|T\|_* \quad \text{subject to } y = \mathcal{B}(T), \quad (2.7)$$

which favors a low-rank solution. To establish recoverability, we can construct a dual certificate  $Q$  of the form  $\mathcal{B}^*(\lambda)$ , whose corresponding dual polynomial satisfies the BIP condition. Here  $\mathcal{B}^*$  is the adjoint operator of  $\mathcal{B}$ . When the operator  $\mathcal{B}$  is random, the concentration of measure guarantees that one can construct a dual certificate  $\mathcal{B}^*(\lambda)$  that is close to the one constructed in the full data case. This fact can then be exploited to verify the BIP property of  $\mathcal{B}^*(\lambda)$  and to establish exact recovery. When the atoms are complex exponentials parameterized by continuous frequencies, this strategy is adopted to establish the compressed sensing off the grid result (the completion problem) [20] building upon the dual polynomial constructed for the super-resolution problem (the full data case) [19]. It shows that the number of random linear measurements required for exact recovery approaches the information theoretical limit. In addition to exact recovery from noise-free measurements, the dual certificate for the full data case can also be utilized to derive near-minimax denoising performance [24, 25], approximate support recovery [25, 26, 27], and robust recovery from observations corrupted by outliers [28]. We expect that the dual polynomial constructed for tensor decomposition will play a similar role for tensor inverse problems, enabling the development of tensor results parallel to their matrix counterparts such as matrix completion, denoising, and robust principal component analysis. We leave these to future work.

## 3 Computational Methods

Our major theorem shows that when the tensor factors satisfies the Assumptions I, II, III and if  $r \leq \frac{n^{17/16}}{12c^2 \sqrt{\tau(\log n)}}$ , we can recover  $\mu^*$  (hence the rank-one factors  $\{(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)\}_{p=1}^r$ ) by solving a convex, infinite-dimensional optimization (1.4). However, as a measure optimization problem, the optimization (1.4) is not directly solvable on a computer. In this section, we discuss two computational methods based respectively on Burer-Montorio's low-rank factorization idea [29] and the Sum-of-Squares relaxations [30, 31].

Our first method is based on factorizing  $T = \sum_{p=1}^{\tilde{r}} \mathbf{u}_p \otimes \mathbf{v}_p \otimes \mathbf{w}_p$  when an upper bound  $\tilde{r}$  on  $r$  is known:

**Proposition 3.1.** *Suppose the decomposition that achieves the tensor nuclear norm  $\|T\|_*$  involves  $r$  terms and  $\tilde{r} \geq r$ , then  $\|T\|_*$  is equal to the optimal value of the following optimization:*

$$\begin{aligned} & \underset{\{(\mathbf{u}_p, \mathbf{v}_p, \mathbf{w}_p)\}_{p=1}^{\tilde{r}}}{\text{minimize}} \quad \frac{1}{3} \left( \sum_{p=1}^{\tilde{r}} [\|\mathbf{u}_p\|_2^3 + \|\mathbf{v}_p\|_2^3 + \|\mathbf{w}_p\|_2^3] \right) \\ & \text{subject to } T = \sum_{p=1}^{\tilde{r}} \mathbf{u}_p \otimes \mathbf{v}_p \otimes \mathbf{w}_p \end{aligned} \quad (3.1)$$

*Proof.* Suppose the tensor nuclear norm is achieved by the following decomposition

$$T = \sum_{p=1}^r \lambda_p^* \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^*.$$



Clearly  $\{(\lambda_p^{*1/3} \mathbf{u}_p^*, \lambda_p^{*1/3} \mathbf{v}_p^*, \lambda_p^{*1/3} \mathbf{w}_p^*)\}_{p=1}^{\tilde{r}}$  forms a feasible solution to (3.1) when  $\tilde{r} = r$ . When  $\tilde{r} > r$ , we can zero-pad the remaining rank-one factors  $\{\mathbf{u}_p, \mathbf{v}_p, \mathbf{w}_p\}_{p=r+1}^{\tilde{r}}$ . The objective function value at this feasible solution is  $\frac{1}{3} \left( \sum_{p=1}^{\tilde{r}} 3\lambda_p^* \right) = \|T\|_*$ . This shows that the optimal value of (3.1) is less than or equal to  $\|T\|_*$ .

To show the other direction, suppose an optimal solution of (3.1) is  $\{(\mathbf{u}_p, \mathbf{v}_p, \mathbf{w}_p)\}_{p=1}^{\tilde{r}}$ . Define

$$a_p = \|\mathbf{u}_p\|_2, b_p = \|\mathbf{v}_p\|_2, c_p = \|\mathbf{w}_p\|_2,$$

and

$$\lambda_p = a_p b_p c_p.$$

For  $p$  such that  $\lambda_p \neq 0$ , define

$$\hat{\mathbf{u}}_p = \mathbf{u}_p/a_p, \hat{\mathbf{v}}_p = \mathbf{v}_p/b_p, \hat{\mathbf{w}}_p = \mathbf{w}_p/c_p.$$

Then clearly

$$T = \sum_{p: \lambda_p \neq 0} \mathbf{u}_p \otimes \mathbf{v}_p \otimes \mathbf{w}_p = \sum_{p=1}^{\tilde{r}} \lambda_p \hat{\mathbf{u}}_p \otimes \hat{\mathbf{v}}_p \otimes \hat{\mathbf{w}}_p.$$

Furthermore, we have

$$\begin{aligned} \|T\|_* \leq \sum_p \lambda_p &= \sum_p a_p b_p c_p \\ &\leq \frac{1}{3} \sum_p (a_p^3 + b_p^3 + c_p^3) \\ &= \frac{1}{3} \sum_{p=1}^{\tilde{r}} [\|\mathbf{u}_p\|_2^3 + \|\mathbf{v}_p\|_2^3 + \|\mathbf{w}_p\|_2^3] \\ &= \text{optimal value of (3.1)}. \end{aligned}$$

Therefore, the optimal value of (3.1) is equal to  $\|T\|_*$ .  $\square$

Proposition 3.1 implies that when an upper bound on  $r$  is known, one can solve the nonlinear (and non-convex) program (3.1) to compute the tensor nuclear norm (and obtain the corresponding decomposition). Numerical simulations suggest that the nonlinear program (3.1), when solved using the ADMM approach [32], has superior performance. Although in theory only local optima can be obtained for the nonlinear programming formulation (3.1), in practice for tensors with randomly generated rank-one factors, the decomposition can almost always be recovered by the ADMM implementation of (3.1).

### 3.1 Sum-of-Squares Relaxations

As a special moment problem, the optimization (1.4) can be approximated increasingly tightly by the semidefinite programs in the Lasserre relaxation hierarchy [30]. The Lasserre hierarchy proposes that instead of optimizing with respect to the measure  $\mu$  in (1.4), one can equivalently optimize the (infinite-dimensional) moment sequence corresponding to  $\mu$ :

$$\mathbf{m} = [m_\alpha] = \int_{\mathbb{K}} \boldsymbol{\xi}^\alpha \mu(d\boldsymbol{\xi}).$$

Here the combined variable  $\boldsymbol{\xi} = [\mathbf{u}^\top \ \mathbf{v}^\top \ \mathbf{w}^\top]^\top \in \mathbb{R}^{3n}$ , the multi-integer index  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{3n})$ , and the monomial  $\boldsymbol{\xi}^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_{3n}^{\alpha_{3n}}$ . To get a finite-dimensional relaxation, we truncate the infinite-dimensional



moment sequence  $\mathbf{m}$  to a finite-dimensional vector  $\mathbf{m}_{2d}$  that includes moments up to order  $2d$ , *i.e.*, to retain moments  $m_{\alpha}$  with  $|\alpha| = \sum_{i=1}^{3n} \alpha_i \leq 2d$ . Three sets of linear matrix inequalities should be satisfied for a vector  $\mathbf{m}_{2d}$  to be the  $2d$ th order truncation of a moment sequence on  $\mathbb{K}$ . First, since the moment matrix here is related with some positive measure  $\mu$ , *i.e.*,

$$M_{2d}(\mathbf{m}_{2d}) := \int_{\mathbb{K}} \begin{bmatrix} 1 \\ \xi_1 \\ \vdots \\ \xi_{3n}^d \end{bmatrix} \begin{bmatrix} 1 \\ \xi_1 \\ \vdots \\ \xi_{3n}^d \end{bmatrix}^{\top} d\mu,$$

it is positive semidefinite. The notation suggests  $M_{2d}(\mathbf{m}_{2d})$  is a (linear) function of the truncated moment vector  $\mathbf{m}_{2d}$ . In addition, since the tensor entries are third order moments of the measure, elements of  $\mathbf{m}_{2d}$  corresponding to these moments are known when  $d \geq 2$ , giving rise to the second set of linear equations. Third, the fact that  $\mu$  is supported on  $\mathbb{K}$  leads to the last set of linear constraints. Combined with the fact that the objective function  $\mu(\mathbb{K}) = \int_{\mathbb{K}} 1 d\mu = \mathbf{m}_{2d}(1)$ , the final relaxation is a semidefinite program. These relaxations are also called sum-of-squares (SOS) relaxations, as in the dual formulation the truncation process is equivalent to replacing positive polynomials with sum-of-squares polynomials [31]. Apparently, increasing the relaxation order  $d$  yields tighter approximations to the original optimization (1.4). The authors of [21] showed that for symmetric tensor decomposition, in the undercomplete case and under a soft-orthogonality condition, the smallest semidefinite program in the relaxation hierarchy is tight. It remains an open question to extend this result to the non-symmetric, over-complete scenario.

## 4 Prior Art and Inspirations

Despite the advantages provided by tensor methods in many applications, their widespread adoption has been slow due to inherent computational intractability. Although the CP decomposition (1.1) is a multi-mode generalization of the singular value decomposition for matrices, determining the CP decomposition for a given tensor is a non-trivial problem that is still under active investigation (cf. [33, 34]). Indeed, even determining the rank of a third-order tensor is an NP-hard problem [13]. A common strategy used to compute a tensor decomposition is to apply an alternating minimization scheme. Although efficient, this approach has the drawback of not providing global convergence guarantees [33]. Recently, an approach combining alternating minimization with power iteration has gained popularity due to its ability to guarantee the tensor decomposition results under certain assumptions [17, 35].

Tensor decomposition is a special case of atomic decomposition which is to determine when a decomposition with respect to some given atomic set  $\mathcal{A}$  achieves the atomic norm. For finite atomic sets, it is now well-known that if the atoms satisfy certain incoherence conditions such as the restricted isometry property, then a sparse decomposition achieves the atomic norm [36]. For the set of rank-one, unit-norm matrices, the atomic norm (the matrix nuclear norm), is achieved by orthogonal decompositions [12]. When the atoms are complex sinusoids parameterized by the frequency, Candès and Fernandez-Granda showed that atomic decomposition is solved by atoms with well-separated frequencies [19]. Similar separation conditions also show up when the atoms are translations of a known waveform [37, 38], spherical harmonics [22], and radar signals parameterized by translations and modulations [39]. The authors of [21] employ the same atomic norm idea but focus on symmetric tensors. In addition, the result of [21] does not apply to overcomplete decompositions. Under a set of conditions, including the incoherence condition ensuring the separation of tensor factors, this work characterizes a class of non-symmetric and overcomplete tensor decompositions that achieve the tensor nuclear norm  $\|T\|_*$ .

Another closely related line of work is matrix completion and tensor completion. Low-rank matrix completion and recovery based on the idea of nuclear norm minimization has received a great deal of attention in recent years [11, 12, 40]. A direct generalization of this approach to tensors would have been using tensor nuclear norm to perform low-rank tensor completion and recovery. However, this approach was not pursued due to the NP-hardness of computing the tensor nuclear norm [13] and the lack of analysis tools for tensor

problems. The mainstream tensor completion approaches are based on various forms of matricization and application of matrix completion to the flattened tensor [41, 1, 42]. Alternating minimization can also be applied to tensor completion and recovery with performance guarantees established in recent work [43]. Most matricization and alternating minimization approaches do not yield optimal bounds on the number of measurements needed for tensor completion. One exception is [44], which uses a special class of separable sampling schemes.

In contrast, we expect that the atomic norm, when specialized to tensors, will achieve the information theoretical limit for tensor completion as it does for compressive sensing, matrix completion [40], and line spectral estimation with missing data [20]. Given a set of atoms, the atomic norm is an abstraction of  $\ell_1$ -type regularization that favors simple models. Using the notion of descent cones, the authors of [15] argued that the atomic norm is the best possible convex proxy for recovering simple models. Particularly, atomic norms are shown in many problems beyond compressive sensing and matrix completion to be able to recover simple models from minimal number of linear measurements. For example, when specialized to the atomic set formed by complex exponentials, the atomic norm can recover signals having sparse representations in the continuous frequency domain with the number of measurements approaching the information theoretic limit without noise [20], as well as achieving near minimax denoising performance [25]. Continuous frequency estimation using the atomic norm is also an instance of measure estimation from (trigonometric) moments.

## 5 Proof of Theorem 1.1

The proof of Theorem 1.1 relies on the construction of a dual polynomial that certifies the optimality of the decomposition measure  $\mu^*$ . The constructed dual polynomial is also essential to the development of tensor completion and denoising results using the atomic norm approach.

### 5.1 Minimal Energy Dual Certificate Construction

According to Proposition 2.1, to prove Theorem 1.1, it suffices to construct a dual polynomial of the form (2.3) that satisfies the BIP. Since the BIP is hard to enforce directly, we start from a pre-certificate that satisfies the following weaker conditions:

$$\begin{aligned} q(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*) &= 1, p = 1, \dots, r \text{ (Interpolation),} \\ q(\mathbf{u}, \mathbf{v}, \mathbf{w}) &\text{ achieves maximum at } (\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*) \text{ on } \mathbb{K}, p = 1, \dots, r \text{ (Maximum).} \end{aligned}$$

Apparently, the Maximum condition is necessary for the Boundedness condition, but generally not sufficient. To satisfy the Boundedness condition, we further minimize the energy  $\|Q\|_F^2 = \sum_{ijk} Q_{ijk}^2$  in the hope that this will push  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  towards zero. Therefore, our candidate dual certificate is the solution to

$$\begin{aligned} &\underset{Q}{\text{minimize}} \quad \|Q\|_F^2 \\ &\text{subject to} \quad q(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*) = 1, p = 1, \dots, r \\ &\quad \quad \quad q(\mathbf{u}, \mathbf{v}, \mathbf{w}) \text{ achieves maximum at } \{(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)\}_{p=1}^r \text{ on } \mathbb{K}. \end{aligned}$$

We next further relax the Maximum condition to a set of linear constraints obtained from the KKT necessary conditions for local maxima. These relaxed conditions also subsume the interpolation condition.

**Lemma 5.1.** *The following conditions are necessary for the Interpolation and Maximum conditions:*

$$\begin{aligned} \sum_{j,k} Q_{ijk} \mathbf{v}_p^*(j) \mathbf{w}_p^*(k) &= \mathbf{u}_p^*(i), i = 1, \dots, n; p = 1, \dots, r; \\ \sum_{i,k} Q_{ijk} \mathbf{u}_p^*(i) \mathbf{w}_p^*(k) &= \mathbf{v}_p^*(j), j = 1, \dots, n; p = 1, \dots, r; \\ \sum_{i,j} Q_{ijk} \mathbf{u}_p^*(i) \mathbf{v}_p^*(j) &= \mathbf{w}_p^*(k), k = 1, \dots, n; p = 1, \dots, r. \end{aligned} \tag{5.1}$$

Thus, we propose solving the following *linear*, minimum norm problem to obtain a dual certificate candidate:

$$\underset{Q}{\text{minimize}} \quad \|Q\|_F^2 \quad \text{subject to} \quad (5.1). \quad (5.2)$$

**Lemma 5.2** (Explicit Form of the Dual Certificate Candidate). *The solution of the least-norm problem (5.2) has the form (normal equations):*

$$Q = \sum_{p=1}^r (\alpha_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \beta_p^* \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \gamma_p^*)$$

where the unknown coefficients  $\{\alpha_p^*, \beta_p^*, \gamma_p^*\}_{p=1}^r$  can be obtained by requiring  $Q$  to satisfy the linear constraints in (5.1). In particular, the candidate dual polynomial has the form

$$\begin{aligned} q(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \langle Q, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \rangle \\ &= \sum_{p=1}^r [\langle \alpha_p^*, \mathbf{u} \rangle \langle \mathbf{v}_p^*, \mathbf{v} \rangle \langle \mathbf{w}_p^*, \mathbf{w} \rangle + \langle \mathbf{u}_p^*, \mathbf{u} \rangle \langle \beta_p^*, \mathbf{v} \rangle \langle \mathbf{w}_p^*, \mathbf{w} \rangle + \langle \mathbf{u}_p^*, \mathbf{u} \rangle \langle \mathbf{v}_p^*, \mathbf{v} \rangle \langle \gamma_p^*, \mathbf{w} \rangle]. \end{aligned} \quad (5.3)$$

In the following, we prove that the candidate dual polynomial given in Lemma 5.2 is a valid dual polynomial under Assumptions I, II, and III.

**Proposition 5.1.** *Under Assumptions I, II, III and if  $r \leq \frac{n^{17/16}}{12c^2\sqrt{\tau(\log n)}}$ , then for sufficiently large  $n$ , the constructed  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  satisfies the Bounded Interpolation Conditions.*

## 5.2 Proof of Proposition 5.1

In order to prove Proposition 5.1, we should check that  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  satisfies the Interpolation condition and the Boundedness condition. The interpolation property is a consequence of the construction process. Hence, it remains to show

$$q(\mathbf{u}, \mathbf{v}, \mathbf{w}) < 1, \forall (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{K}/S.$$

For this purpose, we check  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  in two regions: the “far-away” region that is not close to any support point  $(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)$  and the “near” region that is close to one support point. These two regions together cover the entire set  $\mathbb{K}$ .

Since the explicit form of  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  is determined by the coefficients  $\{\alpha_p^*, \beta_p^*, \gamma_p^*\}_{p=1}^r$ , we need to estimate their values. To get some intuition, let us first compute the value of  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  at  $(\mathbf{u}_i^*, \mathbf{v}_i^*, \mathbf{w}_i^*)$  for some  $i \in [r]$

$$q(\mathbf{u}_i^*, \mathbf{v}_i^*, \mathbf{w}_i^*) = \sum_{p=1}^r [\langle \alpha_p^*, \mathbf{u}_i^* \rangle \langle \mathbf{v}_p^*, \mathbf{v}_i^* \rangle \langle \mathbf{w}_p^*, \mathbf{w}_i^* \rangle + \langle \mathbf{u}_p^*, \mathbf{u}_i^* \rangle \langle \beta_p^*, \mathbf{v}_i^* \rangle \langle \mathbf{w}_p^*, \mathbf{w}_i^* \rangle + \langle \mathbf{u}_p^*, \mathbf{u}_i^* \rangle \langle \mathbf{v}_p^*, \mathbf{v}_i^* \rangle \langle \gamma_p^*, \mathbf{w}_i^* \rangle]. \quad (5.4)$$

Then, consider the simple case where  $U = [\mathbf{u}_1^* \cdots \mathbf{u}_r^*]$ ,  $V = [\mathbf{v}_1^* \cdots \mathbf{v}_r^*]$ ,  $W = [\mathbf{w}_1^* \cdots \mathbf{w}_r^*]$  are almost orthogonal. This allows us to simplify (5.4) as

$$q(\mathbf{u}_i^*, \mathbf{v}_i^*, \mathbf{w}_i^*) \approx \langle \alpha_i^*, \mathbf{u}_i^* \rangle + \langle \beta_i^*, \mathbf{v}_i^* \rangle + \langle \gamma_i^*, \mathbf{w}_i^* \rangle. \quad (5.5)$$

Combining (5.5) with the interpolation condition, we obtain

$$\alpha_i^* \approx \frac{1}{3} \mathbf{u}_i^*, \quad \beta_i^* \approx \frac{1}{3} \mathbf{v}_i^*, \quad \gamma_i^* \approx \frac{1}{3} \mathbf{w}_i^*.$$

This implies the coefficients  $\{\alpha_p^*, \beta_p^*, \gamma_p^*\}_{p=1}^r$  are approximately scaled versions of the tensor factors  $\{\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*\}_{p=1}^r$ . For more general cases, the following lemma provides a tight bound on the differences between the coefficients and the scaled versions of the respective tensor factors.

**Lemma 5.3** (Coefficients Estimates). *Under Assumptions II and III, if  $r \ll \frac{n^2}{\kappa(\log n)^2}$ , the following estimates are valid for sufficiently large  $n$ :*

$$\begin{aligned}\left\|A^* - \frac{1}{3}U\right\| &\leq 2\kappa(\log n) \left(\frac{\sqrt{r}}{n} + c\frac{r}{n^{1.5}}\right); \\ \left\|B^* - \frac{1}{3}V\right\| &\leq 2\kappa(\log n) \left(\frac{\sqrt{r}}{n} + c\frac{r}{n^{1.5}}\right); \\ \left\|C^* - \frac{1}{3}W\right\| &\leq 2\kappa(\log n) \left(\frac{\sqrt{r}}{n} + c\frac{r}{n^{1.5}}\right)\end{aligned}$$

where

$$\begin{aligned}A^* &= [\alpha_1^*, \dots, \alpha_r^*], B^* = [\beta_1^*, \dots, \beta_r^*], C^* = [\gamma_1^*, \dots, \gamma_r^*] \\ U &= [\mathbf{u}_1^*, \dots, \mathbf{u}_r^*], V = [\mathbf{v}_1^*, \dots, \mathbf{v}_r^*], W = [\mathbf{w}_1^*, \dots, \mathbf{w}_r^*]\end{aligned}$$

and the norm  $\|\cdot\|$  is the matrix spectral norm.

Next, we bound  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  using Lemma 5.3 in the far-away region. We define the far-away region more precisely as

$$\mathcal{F}(d) := \{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{K} : |\langle \mathbf{u}, \mathbf{u}_p^* \rangle| \leq d, |\langle \mathbf{v}, \mathbf{v}_p^* \rangle| \leq d, |\langle \mathbf{w}, \mathbf{w}_p^* \rangle| \leq d, \forall p\}$$

with the parameter  $d$  to be determined later. For  $n = 3$  and  $r = 2$ , the far-away region projected onto the unit sphere  $\{\mathbf{u} : \|\mathbf{u}\|_2 = 1\}$  is shown in Figure 1.

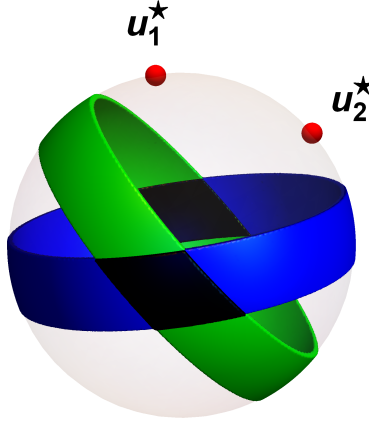


Figure 1: Projection of the far-away region in the  $\mathbf{u}$  coordinates. The blue band represents the region that is far away from  $\mathbf{u}_1^*$ , while the green region is associated with  $\mathbf{u}_2^*$ . The far-away region is their intersection, the black region.

The following lemma bounds  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  in the far-away region:

**Lemma 5.4** (Far-away Region Guarantee). *Under Assumptions II, III and additionally if  $r \ll \frac{n^2}{\kappa(\log n)^2}$ , then the dual polynomial  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  is bounded in the far-away region  $\mathcal{F}(d)$ :*

$$q(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq \left[ d + 6\kappa(\log n) \left( \frac{\sqrt{r}}{n} + c\frac{r}{n^{1.5}} \right) \right] \left( 1 + c\sqrt{\frac{r}{n}} \right)^2. \quad (5.6)$$

Later we will choose appropriate  $d$  to ensure that the dual polynomial  $q(\mathbf{u}, \mathbf{v}, \mathbf{w}) < 1$  in the far-away region  $\mathcal{F}(d)$ .

In the near region that is close to some  $(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)$ , we need to show that the dual polynomial  $q(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq 1$  with equality only if  $(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)$ . Without loss of generality, we consider  $p = 1$  and focus on the region close to  $(\mathbf{u}_1^*, \mathbf{v}_1^*, \mathbf{w}_1^*)$ . Pick  $(\mathbf{x}, \mathbf{y}, \mathbf{z})$  such that  $\mathbf{x} \perp \mathbf{u}_1^*, \mathbf{y} \perp \mathbf{v}_1^*, \mathbf{z} \perp \mathbf{w}_1^*$  and consider the parameterized points

$$(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) \in \mathbb{K}$$

with

$$\begin{aligned} \mathbf{u}(\theta_1) &= \mathbf{u}_1^* \cos(\theta_1) + \mathbf{x} \sin(\theta_1), \\ \mathbf{v}(\theta_2) &= \mathbf{v}_1^* \cos(\theta_2) + \mathbf{y} \sin(\theta_2), \\ \mathbf{w}(\theta_3) &= \mathbf{w}_1^* \cos(\theta_3) + \mathbf{z} \sin(\theta_3). \end{aligned} \tag{5.7}$$

When  $\theta_1$  ranges from 0 to  $\pi$ ,  $\mathbf{u}(\theta_1)$  traces out a 2D semi-circle that starts at  $\mathbf{u}_1^*$ , passes through  $\mathbf{x}$ , and finally reaches  $-\mathbf{u}_1^*$ . The same properties hold for  $\mathbf{v}(\theta_2)$  and  $\mathbf{w}(\theta_3)$ . In fact, the entire region  $\mathbb{K}$  can be represented as  $\bigcup_{(\mathbf{x}, \mathbf{y}, \mathbf{z}) : \mathbf{x} \perp \mathbf{u}_1^*, \mathbf{y} \perp \mathbf{v}_1^*, \mathbf{z} \perp \mathbf{w}_1^*} \{(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) : \theta_i \in [0, \pi], i = 1, 2, 3\}$ . This parameterization projected onto the  $\mathbf{u}$  coordinates is shown in Figure 2.

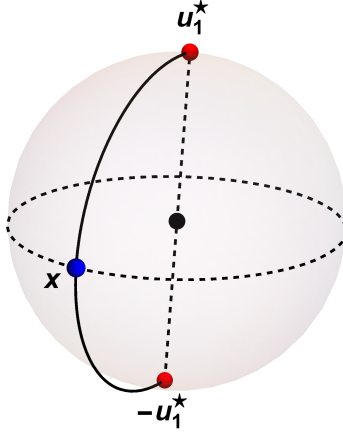


Figure 2: Parameterization of points on the unit sphere for  $\mathbf{u}$ .

**Lemma 5.5** (Parameterized Form of the Dual Polynomial Candidate). *The candidate dual polynomial has the parameterized form*

$$\begin{aligned} q(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) &= \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) \\ &\quad + q(\mathbf{u}_1^*, \mathbf{y}, \mathbf{z}) \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \\ &\quad + q(\mathbf{x}, \mathbf{v}_1^*, \mathbf{z}) \sin(\theta_1) \cos(\theta_2) \sin(\theta_3) \\ &\quad + q(\mathbf{x}, \mathbf{y}, \mathbf{w}_1^*) \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) \\ &\quad + q(\mathbf{x}, \mathbf{y}, \mathbf{z}) \sin(\theta_1) \sin(\theta_2) \sin(\theta_3). \end{aligned} \tag{5.8}$$

Furthermore, under Assumptions I, II, III and if  $r \leq n^{1.25-1.5r_c}$  with  $r_c \in (0, \frac{1}{6})$ , the following estimates are valid for sufficiently large  $n$ :

$$\max\{q(\mathbf{u}_1^*, \mathbf{y}, \mathbf{z}), q(\mathbf{x}, \mathbf{v}_1^*, \mathbf{z}), q(\mathbf{x}, \mathbf{y}, \mathbf{w}_1^*)\} = o(n^{-r_c}) \tag{5.9}$$

$$q(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq 1 + 2\tau(\log n)n^{-r_c} + o(n^{-r_c}). \tag{5.10}$$

which imply that

$$q(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) \leq \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) + \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) + 4\tau(\log n)n^{-r_c}. \quad (5.11)$$

Next, we control the dual polynomial  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  in the near region using Lemma 5.5. For this purpose, we first define the  $i$ th near-region as

$$\mathcal{N}_i(\delta) := \{(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbb{K} : |\langle \mathbf{u}_i^*, \mathbf{u} \rangle| \geq \delta, |\langle \mathbf{v}_i^*, \mathbf{v} \rangle| \geq \delta, |\langle \mathbf{w}_i^*, \mathbf{w} \rangle| \geq \delta\}.$$

The near region  $\mathcal{N}_1(\delta)$  projected onto the sphere  $\{\mathbf{u} : \|\mathbf{u}\|_2 = 1\}$  is shown in Figure 3.

We have the following result for the behavior of  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  in the near region:

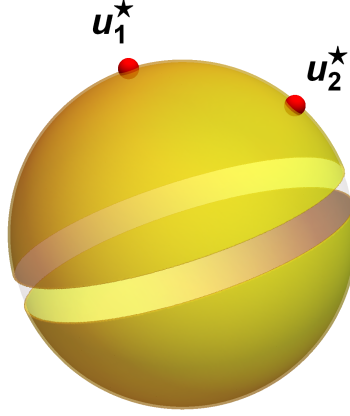


Figure 3: The two yellow spherical caps form the near region  $\mathcal{N}_1(\delta)$  around the point  $(\mathbf{u}_1^*, \mathbf{v}_1^*, \mathbf{w}_1^*)$  projected onto the  $\mathbf{u}$  coordinates.  $\mathcal{N}_2(\delta)$ , which is not shown here, consists of another two spherical caps. The union of  $\mathcal{N}_1(\delta), \mathcal{N}_2(\delta)$  and the far-away region  $\mathcal{F}(d)$  shown in Figure 1 will cover the entire sphere if  $\delta \leq d$ .

**Lemma 5.6** (Near Region Guarantee). *Under Assumptions I, II, III and if  $r \leq n^{1.25-1.5r_c}$  and  $\delta = \sqrt{\frac{80}{3}\tau(\log n)n^{-0.5r_c}}$  with  $r_c \in (0, \frac{1}{6})$ , then the dual polynomial  $q(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq 1$  in  $\mathcal{N}_i(\delta)$  with equality only if  $(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u}_i^*, \mathbf{v}_i^*, \mathbf{w}_i^*)$ .*

The proof is given in Section F. We outline the main idea here. Without loss of generality, we control  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  in  $\mathcal{N}_1(\delta)$ . With the parameterization (5.7) in mind, we recognize that  $\cos(\theta_1) = \langle \mathbf{u}_1^*, \mathbf{u}(\theta_1) \rangle$ , etc., and further restrict us to

$$\mathcal{N}_1(\delta; \mathbf{x}, \mathbf{y}, \mathbf{z}) := \{(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) \in \mathbb{K} : |\cos(\theta_1)| \geq \delta, |\cos(\theta_2)| \geq \delta, |\cos(\theta_3)| \geq \delta\}.$$

Note that  $\mathcal{N}_1(\delta) = \bigcup_{\mathbf{x} \perp \mathbf{u}_1^*, \mathbf{y} \perp \mathbf{v}_1^*, \mathbf{z} \perp \mathbf{w}_1^*} \mathcal{N}_1(\delta; \mathbf{x}, \mathbf{y}, \mathbf{z})$ . We first apply Taylor expansion to the parameterised form (5.8) over the region  $(0, \theta_0] \times (0, \theta_0] \times (0, \theta_0]$  with some appropriate  $\theta_0$  chosen wisely to ensure  $q(\mathbf{u}, \mathbf{v}, \mathbf{w}) < 1$  in this region. Second, we use the upper bound (5.11) to delineate regions of intermediate  $\theta_i$  values where  $q(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) < 1$ . Finally, we combine these two regions to ensure that the function  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  is controlled in  $\mathcal{N}_1(\delta; \mathbf{x}, \mathbf{y}, \mathbf{z})$ . Due to the arbitrariness of  $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ , these imply that  $q(\mathbf{u}, \mathbf{v}, \mathbf{w}) < 1$  in the entire  $\mathcal{N}_1(\delta)$  except at  $(\mathbf{u}_1^*, \mathbf{v}_1^*, \mathbf{w}_1^*)$ .

We are ready to combine the near region and far-away region analyses to prove Proposition 5.1.

*Proof.* Choose  $\gamma = 6\sqrt{\tau(\log n)}$  and  $d = \gamma n^{-r_c/2} - 6\kappa(\log n)(\sqrt{r}/n + crn^{-1.5})$ . The conditions  $r \leq n^{1.25-1.5r_c}$  and  $r_c \in (0, 1/6)$  ensure that  $d > 0$  for sufficiently large  $n$ . Using this value of  $d$  in Lemma 5.4, we have that

for any  $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathcal{F}(d)$ ,

$$\begin{aligned} q(\mathbf{u}, \mathbf{v}, \mathbf{w}) &\leq \left[ d + 6\kappa(\log n) \left( \frac{\sqrt{r}}{n} + c \frac{r}{n^{1.5}} \right) \right] \left( 1 + c \sqrt{\frac{r}{n}} \right)^2 \\ &= \gamma n^{-r_c/2} \left( 1 + 2c \sqrt{\frac{r}{n}} + c^2 \frac{r}{n} \right). \end{aligned}$$

Therefore, to control  $q(\mathbf{u}, \mathbf{v}, \mathbf{w}) < 1$  in  $\mathcal{F}(d)$ , it suffices to require

$$\gamma n^{-r_c/2} < \frac{1}{4}, \quad (5.12)$$

$$c \gamma n^{-r_c/2} \sqrt{\frac{r}{n}} \leq \frac{1}{8}, \quad (5.13)$$

$$c^2 \gamma n^{-r_c/2} \frac{r}{n} \leq \frac{1}{2}. \quad (5.14)$$

First, since  $\gamma n^{-r_c/2} = o(1)$ , (5.12) holds for sufficiently large  $n$ . Second, (5.13) needs  $r \leq \frac{n^{1+r_c}}{64c^2\gamma^2}$  and (5.14) requires  $r \leq \frac{n^{1+r_c/2}}{2c^2\gamma}$ . This can be reduced to  $r \leq \frac{n^{1+r_c/2}}{2c^2\gamma}$  for sufficiently large  $n$ . Therefore, as long as  $r \leq \frac{n^{1+r_c/2}}{2c^2\gamma}$ , we have  $q(\mathbf{u}, \mathbf{v}, \mathbf{w}) < 1$  in  $\mathcal{F}(d)$ .

It remains to ensure that the union of the far-away region and the near region  $\mathcal{F}(d) \cup \bigcup_i \mathcal{N}_i(\delta)$  to be the entire space  $\mathbb{K}$ , which requires  $\delta \leq d$ , that is,

$$\sqrt{\frac{80}{3} \tau(\log n) n^{-r_c/2}} \leq \gamma n^{-r_c/2} - 6\kappa(\log n) (\sqrt{r}/n + crn^{-1.5}).$$

To see that the inequality indeed holds for sufficiently large  $n$ , we first note that

$$6\kappa(\log n) (\sqrt{r}/n + crn^{-1.5}) \ll \gamma n^{-r_c/2}$$

due to  $r \leq n^{1.25-1.5r_c}$ . Hence, we have  $6\kappa(\log n) (\sqrt{r}/n + crn^{-1.5}) \leq (1 - \frac{15}{\sqrt{240}}) \gamma n^{-r_c/2}$ , for sufficiently large  $n$ . Second, we also have

$$\sqrt{\frac{80}{3} \tau(\log n) n^{-r_c/2}} \leq \frac{15}{\sqrt{240}} \gamma n^{-r_c/2},$$

which holds since  $\gamma = 6\sqrt{\tau(\log n)} \geq \frac{80}{15} \sqrt{\tau(\log n)}$ . These ensure that  $\delta \leq d$  for large enough  $n$ .

In conclusion, one needs

$$r \leq n^{1.25-1.5r_c} \text{ and } r \leq \frac{n^{1+0.5r_c}}{2c^2\gamma}. \quad (5.15)$$

By choosing  $r_c = 1/8 \in (0, \frac{1}{6})$  and noting that  $\gamma = 6\sqrt{\tau(\log n)}$ , the above constraints (5.15) become

$$r \leq n^{17/16} \text{ and } r \leq \frac{n^{17/16}}{12c^2 \sqrt{\tau(\log n)}}.$$

Thus, we choose

$$r \leq \frac{n^{17/16}}{12c^2 \sqrt{\tau(\log n)}},$$

so that  $q(\mathbf{u}, \mathbf{v}, \mathbf{w}) < 1$  in  $\mathbb{K}$  except at  $\{(\mathbf{u}_i^*, \mathbf{v}_i^*, \mathbf{w}_i^*)\}_{i=1}^r$ . This completes the proof of Proposition 5.1.  $\square$



## 6 Numerical Experiments

We present numerical results to support our theory and to test the proposed computational methods. In particular, we examine the phase transition curves of the rate of success for three algorithms: i) ADMM implementation of (3.1) with “Good Initialization” (ADMM-G), ii) ADMM with random initialization (ADMM-R) and iii) the SOS relaxation of order  $d = 2$  (SOS-2). ADMM with “Good Initialization” uses the output of the power method developed in [17] as initialization.

The phase transition curves are plotted in Figure 4. In preparing Figure 4, the  $r$  rank-one tensor components  $\{\mathbf{u}_p, \mathbf{v}_p, \mathbf{w}_p\}_{p=1}^r$  were generated following i.i.d. Gaussian distribution, and then each  $\mathbf{u}_p, \mathbf{v}_p, \mathbf{w}_p$  was normalized to have a unit norm. We set the coefficients  $\lambda_p = (1 + \varepsilon_p^2)/2$ , where  $\varepsilon_p$  is chosen from the standard normal distribution, to ensure a minimal coefficient of at least  $1/2$ . We varied the dimension  $n$  and the rank  $r$ . For each fixed  $(r, n)$  pair, 5 instances of the tensor were generated. We then ran the three algorithms for each instance and declared success if i) the recovered truncated moment vector is within  $10^{-3}$  distance of the true moment vector for the SOS method, and ii) the recovered tensor factors are within  $10^{-3}$  distance to the true tensor factors. We used the moment vector criteria for the SOS method because one cannot identify more than  $n$  tensor factors for the  $d = 2$  relaxation. Also, considering the high computational complexity of the SOS method when  $n$  is large, we only set  $n$  range from 2 to 8. The rate of success for each algorithm is the percentage of successful instances.

From Figure 1, we observe that the SOS relaxation with  $d = 2$  is unable to identify more than  $n$  factors. The ADMM method works for  $r$  much larger than  $n$ . In addition, random initialization does not degrade the performance compared with “Good Initialization”.

## 7 Conclusion

By explicitly constructing a dual certificate, we derive conditions for a tensor decomposition to achieve the tensor nuclear norm. This implies that the infinite dimensional measure optimization, which defines the tensor nuclear norm, is able to recover the decomposition under an incoherent condition and two other mild conditions. Computational methods based on low-rank factorization and Sum-of-Squares relaxations are used to solve the measure optimization. Numerical experiments show that the nonlinear programming approach has superior performance. Future work will analyze the observed good performance of the nonlinear programming formulation.

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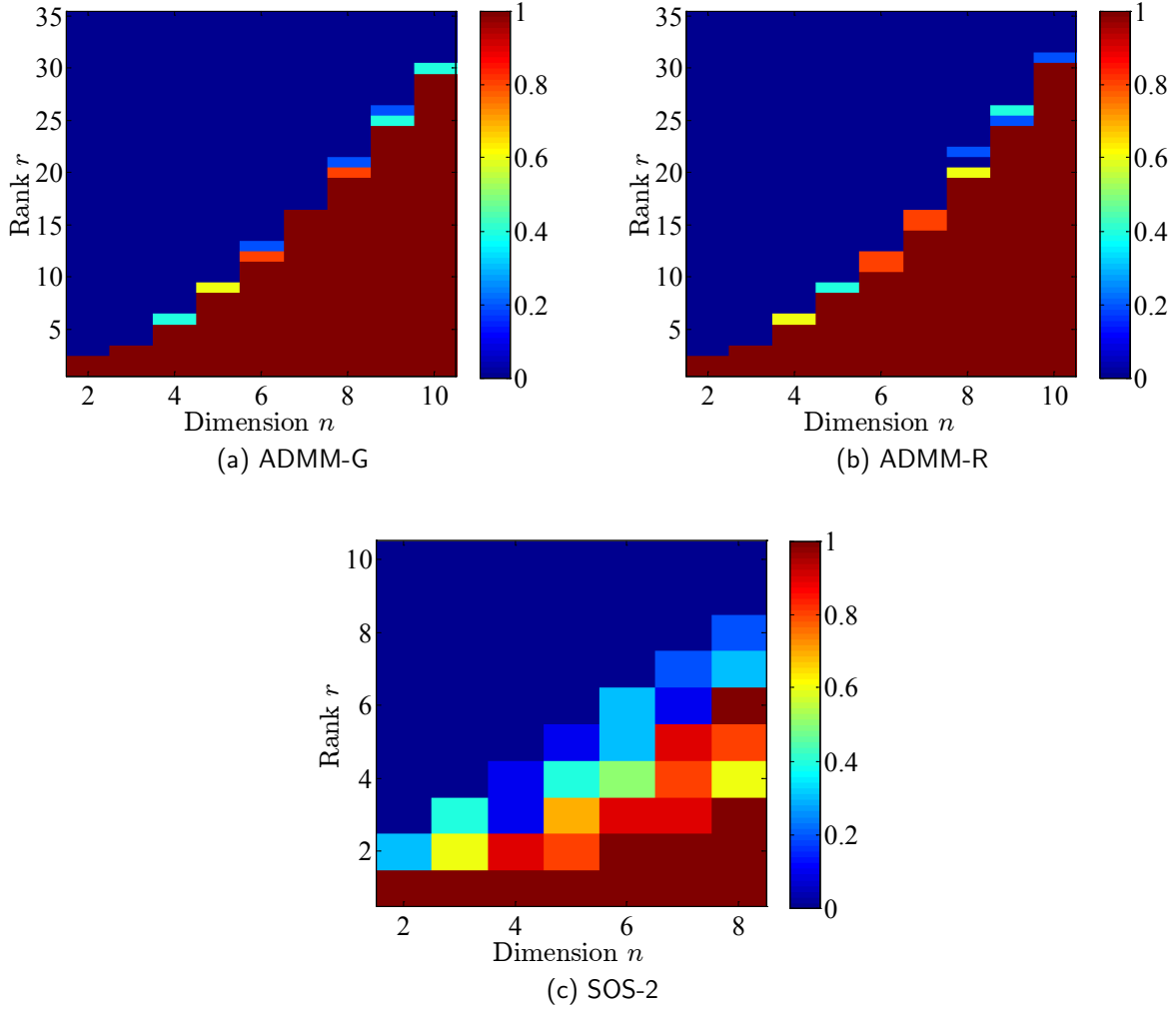


Figure 4: Rate of success for tensor decomposition using ADMM-G, ADMM-R and SOS.

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## A Proof of Lemma 5.1

*Proof.* Since  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  achieves maximum at  $\{(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)\}_{p=1}^r$  on  $\mathbb{K} = \{(\mathbf{u}, \mathbf{v}, \mathbf{w}) : \|\mathbf{u}\|_2^2 = \|\mathbf{v}\|_2^2 = \|\mathbf{w}\|_2^2 = 1\}$ , the gradient of the Lagrangian

$$L(\mathbf{u}, \mathbf{v}, \mathbf{w}) = q(\mathbf{u}, \mathbf{v}, \mathbf{w}) - a(\|\mathbf{u}\|_2^2 - 1) - b(\|\mathbf{v}\|_2^2 - 1) - c(\|\mathbf{w}\|_2^2 - 1)$$

must vanish at  $\{(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)\}_{p=1}^r$ :

$$\begin{aligned} \frac{\partial L(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)}{\partial \mathbf{u}} &= \frac{\partial q}{\partial \mathbf{u}}(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*) - 2a\mathbf{u}_p^* = 0; p = 1, \dots, r; \\ \frac{\partial L(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)}{\partial \mathbf{v}} &= \frac{\partial q}{\partial \mathbf{v}}(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*) - 2b\mathbf{v}_p^* = 0; p = 1, \dots, r; \\ \frac{\partial L(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*)}{\partial \mathbf{w}} &= \frac{\partial q}{\partial \mathbf{w}}(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*) - 2c\mathbf{w}_p^* = 0; p = 1, \dots, r. \end{aligned}$$

Using the fact that

$$\frac{\partial}{\partial \mathbf{u}(i)} q(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{j,k} Q_{ijk} \mathbf{v}(j) \mathbf{w}(k),$$

we obtain

$$\begin{aligned} \sum_{i=1}^n \left[ \frac{\partial}{\partial \mathbf{u}(i)} q(\mathbf{u}, \mathbf{v}, \mathbf{w}) \right] \mathbf{u}_p^*(i) &= \sum_{i,j,k} Q_{ijk} \mathbf{u}_p^*(i) \mathbf{v}_p^*(j) \mathbf{w}_p^*(k) \\ &= q(\mathbf{u}_p^*, \mathbf{v}_p^*, \mathbf{w}_p^*) = 1. \end{aligned}$$

On the other hand, we also have

$$2a \mathbf{u}_p^{*\top} \mathbf{u}_p^* = 2a = 1$$

implying

$$a = \frac{1}{2}.$$

Similar arguments show  $b = c = 1/2$ . Therefore, the following conditions are necessary for the Maximum Condition:

$$\begin{aligned} \sum_{j,k} Q_{ijk} \mathbf{v}_p^*(j) \mathbf{w}_p^*(k) &= \mathbf{u}_p^*(i), i = 1, \dots, n; p = 1, \dots, r; \\ \sum_{i,k} Q_{ijk} \mathbf{u}_p^*(i) \mathbf{w}_p^*(k) &= \mathbf{v}_p^*(j), j = 1, \dots, n; p = 1, \dots, r; \\ \sum_{j,k} Q_{ijk} \mathbf{u}_p^*(i) \mathbf{v}_p^*(j) &= \mathbf{w}_p^*(k), k = 1, \dots, n; p = 1, \dots, r. \end{aligned}$$

These conditions apparently subsume the Interpolation Condition. □

## B Proof of Lemma 5.2

*Proof.* If we denote the linear operator that describes the left-hand side operations on  $Q$  in (5.1) as  $\mathcal{A} : \mathbb{R}^{n \times n \times n} \mapsto \mathbb{R}^{(n+n+n)r}$ , and the right-hand side constant vector as  $\mathbf{s}$ , then the minimum norm problem (5.2) whose optimal point is a dual certificate candidate becomes:

$$\text{minimize } \|Q\|_F^2 \quad \text{subject to } \mathcal{A}(Q) = \mathbf{s}.$$

The solution of the above least-norm problem is given by

$$Q = \mathcal{A}^* \begin{bmatrix} \boldsymbol{\alpha}^* \\ \boldsymbol{\beta}^* \\ \boldsymbol{\gamma}^* \end{bmatrix}$$

where  $\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*$  satisfies the normal equation:

$$\mathcal{A} \mathcal{A}^* \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{bmatrix} = \mathbf{s}.$$

Since

$$\begin{aligned}
\left\langle \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{bmatrix}, \mathcal{A}(Q) \right\rangle &= \sum_{p=1}^r \sum_i \alpha_p(i) \sum_{j,k} Q_{ijk} \mathbf{v}_p^*(j) \mathbf{w}_p^*(k) + \sum_{p=1}^r \sum_j \beta_p(j) \sum_{i,k} Q_{ijk} \mathbf{u}_p^*(i) \mathbf{w}_p^*(k) \\
&\quad + \sum_{p=1}^r \sum_k \gamma_p(k) \sum_{i,j} Q_{ijk} \mathbf{u}_p^*(i) \mathbf{v}_p^*(j) \\
&= \sum_{i,j,k} Q_{ijk} \sum_{p=1}^r [\alpha_p(i) \mathbf{v}_p^*(j) \mathbf{w}_p^*(k) + \mathbf{u}_p^*(i) \beta_p(j) \mathbf{w}_p^*(k) + \mathbf{u}_p^*(i) \mathbf{v}_p^*(j) \gamma_p(k)] \\
&= \left\langle \sum_{p=1}^r \alpha_p \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \beta_p \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \gamma_p, Q \right\rangle \\
&= \left\langle \mathcal{A}^* \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{bmatrix}, Q \right\rangle,
\end{aligned}$$

we obtain the following explicit formula for  $\mathcal{A}^*$ :

$$\mathcal{A}^* \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{bmatrix} = \sum_{p=1}^r (\alpha_p \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \beta_p \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \gamma_p).$$

Plugging  $Q = \mathcal{A}^* \begin{bmatrix} \boldsymbol{\alpha}^* \\ \boldsymbol{\beta}^* \\ \boldsymbol{\gamma}^* \end{bmatrix}$  into  $q(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \langle Q, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \rangle$ , we get that the minimal energy dual polynomial has the form

$$\begin{aligned}
q(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \langle Q, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \rangle \\
&= \sum_{p=1}^r \langle \alpha_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \beta_p^* \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \gamma_p^*, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \rangle \\
&= \sum_{p=1}^r [\langle \alpha_p^*, \mathbf{u} \rangle \langle \mathbf{v}_p^*, \mathbf{v} \rangle \langle \mathbf{w}_p^*, \mathbf{w} \rangle + \langle \mathbf{u}_p^*, \mathbf{u} \rangle \langle \beta_p^*, \mathbf{v} \rangle \langle \mathbf{w}_p^*, \mathbf{w} \rangle + \langle \mathbf{u}_p^*, \mathbf{u} \rangle \langle \mathbf{v}_p^*, \mathbf{v} \rangle \langle \gamma_p^*, \mathbf{w} \rangle].
\end{aligned}$$

□

## C Proof of Lemma 5.3

*Proof.* We need to find coefficient vectors  $\{\boldsymbol{\alpha}_p^*, \boldsymbol{\beta}_p^*, \boldsymbol{\gamma}_p^*\}_{p=1}^r$  such that the tensor

$$Q = \sum_{p=1}^r (\alpha_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \beta_p^* \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \gamma_p^*)$$

satisfies

$$\begin{aligned}
\sum_{j,k} Q_{ijk} \mathbf{v}_p^*(j) \mathbf{w}_p^*(k) &= \mathbf{u}_p^*(i), i = 1, \dots, n; p = 1, \dots, r; \\
\sum_{i,k} Q_{ijk} \mathbf{u}_p^*(i) \mathbf{w}_p^*(k) &= \mathbf{v}_p^*(j), j = 1, \dots, n; p = 1, \dots, r; \\
\sum_{j,k} Q_{ijk} \mathbf{u}_p^*(i) \mathbf{v}_p^*(j) &= \mathbf{w}_p^*(k), k = 1, \dots, n; p = 1, \dots, r,
\end{aligned}$$

or in tensor notation

$$\begin{aligned}
Q(I, \mathbf{v}_p^*, \mathbf{w}_p^*) &:= Q \bar{\times}_2 \mathbf{v}_p^* \bar{\times}_3 \mathbf{w}_p^* = \mathbf{u}_p^*, p = 1, \dots, r; \\
Q(\mathbf{u}_p^*, I, \mathbf{w}_p^*) &:= Q \bar{\times}_1 \mathbf{u}_p^* \bar{\times}_3 \mathbf{w}_p^* = \mathbf{v}_p^*, p = 1, \dots, r; \\
Q(\mathbf{u}_p^*, \mathbf{v}_p^*, I) &:= Q \bar{\times}_1 \mathbf{u}_p^* \bar{\times}_2 \mathbf{v}_p^* = \mathbf{w}_p^*, p = 1, \dots, r,
\end{aligned} \tag{C.1}$$

where the definitions of the tensor-vector products  $\{\bar{\times}_i\}$  are apparent from context. When the expression of  $Q$  is plugged into (C.1), we obtain that the coefficients  $\{\alpha_p^*, \beta_p^*, \gamma_p^*\}_{p=1}^r$  should satisfy the normal equation

$$\mathcal{A}\mathcal{A}^* \begin{bmatrix} \alpha^* \\ \beta^* \\ \gamma^* \end{bmatrix} = s.$$

The operator  $\mathcal{A}\mathcal{A}^*$  is a  $3n \times 3n$  matrix with rank  $(3n - 2)r$ , whose null space of dimension  $2r$  is given by

$$\text{span} \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \mathbf{u}_p^* \otimes \mathbf{e}_p \\ -\frac{1}{\sqrt{2}} \mathbf{v}_p^* \otimes \mathbf{e}_p \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \mathbf{u}_p^* \otimes \mathbf{e}_p \\ \frac{1}{\sqrt{6}} \mathbf{v}_p^* \otimes \mathbf{e}_p \\ -\frac{2}{\sqrt{6}} \mathbf{w}_p^* \otimes \mathbf{e}_p \end{pmatrix}, p = 1, \dots, r \right\}.$$

Here  $\mathbf{e}_p$  is the  $p$ th canonical basis vector. In the following, we will take  $\begin{bmatrix} \alpha^* \\ \beta^* \\ \gamma^* \end{bmatrix} = (\mathcal{A}\mathcal{A}^*)^\dagger s$ .

For analysis purpose, we alternatively solve for  $\{\alpha_p^*, \beta_p^*, \gamma_p^*\}_{p=1}^r$  iteratively using the following iterations

$$\begin{aligned}
\alpha_q^{t+1} &= \alpha_q^t - \rho(Q_1^t(I, \mathbf{v}_q^*, \mathbf{w}_q^*) - \mathbf{u}_q^*), q = 1, \dots, r; \\
\beta_q^{t+1} &= \beta_q^t - \rho(Q_2^t(\mathbf{u}_q^*, I, \mathbf{w}_q^*) - \mathbf{v}_q^*), q = 1, \dots, r; \\
\gamma_q^{t+1} &= \gamma_q^t - \rho(Q_3^t(\mathbf{u}_q^*, \mathbf{v}_q^*, I) - \mathbf{w}_q^*), q = 1, \dots, r,
\end{aligned}$$

initialized by

$$\begin{aligned}
\alpha_q^0 &= \frac{1}{3} \mathbf{u}_q^*, q = 1, \dots, r; \\
\beta_q^0 &= \frac{1}{3} \mathbf{v}_q^*, q = 1, \dots, r; \\
\gamma_q^0 &= \frac{1}{3} \mathbf{w}_q^*, q = 1, \dots, r.
\end{aligned}$$

Here  $\rho$  is a step size parameter to be chosen later and the tensors

$$\begin{aligned}
Q_1^t &:= \sum_{p=1}^r (\alpha_p^t \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \beta_p^t \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \gamma_p^t); \\
Q_2^t &:= \sum_{p=1}^r (\alpha_p^t \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \beta_p^t \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \gamma_p^t); \\
Q_3^t &:= \sum_{p=1}^r (\alpha_p^t \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \beta_p^t \otimes \mathbf{w}_p^* + \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \gamma_p^t).
\end{aligned}$$

Note the differences in the definitions of  $Q_1^t, Q_2^t$  and  $Q_3^t$ . Also note that the iterative procedures are for theoretical analysis only. Therefore, we used  $\{\alpha_p^*, \beta_p^*, \gamma_p^*\}_{p=1}^r$  in the definitions of  $Q_1^t, Q_2^t$  and  $Q_3^t$ .



We next establish the convergence of the iterations. Plugging the tensor eigenvalue equations (C.1) and subtracting from both sides of the iterations the true solution values yield

$$\begin{aligned} & \alpha_q^{t+1} - \alpha_q^* \\ &= \alpha_q^t - \alpha_q^* - \rho[Q_1^t(I, \mathbf{v}_q^*, \mathbf{w}_q^*) - Q(I, \mathbf{v}_q^*, \mathbf{w}_q^*)] \\ &= \alpha_q^t - \alpha_q^* - \rho \sum_{p=1}^r [(\alpha_p^t - \alpha_p^*) \langle \mathbf{v}_p^*, \mathbf{v}_q^* \rangle \langle \mathbf{w}_p^*, \mathbf{w}_q^* \rangle + \mathbf{u}_p^* \langle \beta_p^* - \beta_p^*, \mathbf{v}_q^* \rangle \langle \mathbf{w}_p^*, \mathbf{w}_q^* \rangle + \mathbf{u}_p^* \langle \mathbf{v}_p^*, \mathbf{v}_q^* \rangle \langle \gamma_p^* - \gamma_p^*, \mathbf{w}_q^* \rangle]. \end{aligned}$$

We arrange the iterates  $\{\alpha_p^t\}_{p=1}^r$  and the coefficients  $\{\alpha_p^*\}_{p=1}^r$  into an  $n \times r$  matrix by defining

$$\begin{aligned} A^t &:= [\alpha_1^t, \dots, \alpha_r^t], \\ A^* &:= [\alpha_1^*, \dots, \alpha_r^*], \end{aligned}$$

and similarly  $B^t, C^t, B^*, C^*$ . Recognizing

$$\begin{aligned} \sum_{p=1}^r (\alpha_p^t - \alpha_p^*) \langle \mathbf{v}_p^*, \mathbf{v}_q^* \rangle \langle \mathbf{w}_p^*, \mathbf{w}_q^* \rangle &= (A^t - A^*) [(V^\top \mathbf{v}_q^*) \odot (W^\top \mathbf{w}_q^*)]; \\ \sum_{p=1}^r \mathbf{u}_p^* \langle \beta_p^* - \beta_p^*, \mathbf{v}_q^* \rangle \langle \mathbf{w}_p^*, \mathbf{w}_q^* \rangle &= U [(B^* - B^*)^\top \mathbf{v}_q^* \odot (W^\top \mathbf{w}_q^*)]; \\ \sum_{p=1}^r \mathbf{u}_p^* \langle \mathbf{v}_p^*, \mathbf{v}_q^* \rangle \langle \gamma_p^* - \gamma_p^*, \mathbf{w}_q^* \rangle &= U [(V^\top \mathbf{v}_q^*) \odot ((C^* - C^*)^\top \mathbf{w}_q^*)], \end{aligned}$$

we obtain

$$\begin{aligned} A^{t+1} - A^* &= A^t - A^* - \rho \{ (A^t - A^*) [(V^\top V) \odot (W^\top W)] + U [((B^* - B^*)^\top V) \odot (W^\top W)] \\ &\quad + U [(V^\top V) \odot ((C^* - C^*)^\top W)] \} \\ &= (A^t - A^*) (I - \rho [(V^\top V) \odot (W^\top W)]). \end{aligned}$$

Similarly, we have

$$\begin{aligned} B^{t+1} - B^* &= (B^t - B^*) (I - \rho [(U^\top U) \odot (W^\top W)]) - \rho V [((A^t - A^*)^\top U) \odot (W^\top W)]; \\ C^{t+1} - C^* &= (C^t - C^*) (I - \rho [(U^\top U) \odot (V^\top V)]) \\ &\quad - \rho \{ W [((A^t - A^*)^\top U) \odot (V^\top V)] + W [(U^\top U) \odot ((B^t - B^*)^\top V)] \}. \end{aligned}$$

Denoting by  $e_a^t = \|A^t - A^*\|$ ,  $e_b^t = \|B^t - B^*\|$ ,  $e_c^t = \|C^t - C^*\|$  and defining  $\tilde{\rho} := \rho \min\{\lambda_{\min}((V^\top V) \odot (W^\top W)), \lambda_{\min}((U^\top U) \odot (W^\top W)), \lambda_{\min}((U^\top U) \odot (V^\top V))\}$ , we continue with

$$\begin{bmatrix} e_a^{t+1} \\ e_b^{t+1} \\ e_c^{t+1} \end{bmatrix} \leq \begin{bmatrix} 1 - \tilde{\rho} & 0 & 0 \\ \rho \|U\| \|V\| \|W\|^2 & 1 - \tilde{\rho} & 0 \\ \rho \|U\| \|W\| \|V\|^2 & \rho \|U\|^2 \|V\| \|W\| & 1 - \tilde{\rho} \end{bmatrix} \begin{bmatrix} e_a^t \\ e_b^t \\ e_c^t \end{bmatrix}$$

where we have used triangle inequality and properties of spectral norms such as  $\|P \odot Q\| \leq \|P\| \|Q\|$ <sup>1</sup>.

When the step size  $\rho$  is chosen to be sufficiently small, the error sequence is convergent since the eigenvalues of the system matrix share the same value  $\eta = 1 - \hat{\rho} \in (0, 1)$ . Therefore, the errors converge to zeros geometrically. The speed of convergence can be taken as the spectral radius of the system matrix, which is bounded by  $\eta \leq 1 - \rho \left(1 - \frac{\kappa(\log n) \sqrt{r}}{n}\right)$  invoking Assumption III.

<sup>1</sup>Hadamard product  $P \odot Q$  is a principal submatrix of  $P \otimes Q$ , whose singular values are the products of the individual singular values of  $P$  and  $Q$ .

Subtracting the following two consecutive iterations for  $\{A^t\}$ :

$$\begin{aligned} A^{t+1} - A^* &= (A^t - A^*)(I - \rho[(V^\top V) \odot (W^\top W)]), \\ A^t - A^* &= (A^{t-1} - A^*)(I - \rho[(V^\top V) \odot (W^\top W)]), \end{aligned}$$

yields

$$A^{t+1} - A^t = (A^t - A^{t-1})(I - \rho[(V^\top V) \odot (W^\top W)]).$$

Similar manipulations applied to  $\{B^t\}$  and  $\{C^t\}$  lead to

$$\begin{aligned} B^{t+1} - B^t &= (B^t - B^{t-1})(I - \rho[(U^\top U) \odot (W^\top W)]) - \rho V[((A^t - A^{t-1})^\top U) \odot (W^\top W)], \\ C^{t+1} - C^t &= (C^t - C^{t-1})(I - \rho[(U^\top U) \odot (V^\top V)]) \\ &\quad - \rho\{W[((A^t - A^{t-1})^\top U) \odot (V^\top V)] + W[(U^\top U) \odot ((B^t - B^{t-1})^\top V)]\}. \end{aligned}$$

Overload the notations  $e_a^t = \|A^t - A^{t-1}\|$ ,  $e_b^t = \|B^t - B^{t-1}\|$ ,  $e_c^t = \|C^t - C^{t-1}\|$ . Similar to the previous argument, we also obtain that these errors converge to zero geometrically with a rate  $\eta \in (0, 1)$ . So we have  $\|C^t - C^{t-1}\| \leq \eta^{t-1}\|C^1 - C^0\|$ .

Consequently, we bound the distance between  $C^t$  and  $C^0$  as follows:

$$\begin{aligned} \|C^t - C^0\| &\leq \sum_{s=0}^{t-1} \|C^{s+1} - C^s\| \\ &\leq \sum_{s=0}^{t-1} \eta^s \|C^1 - C^0\| \\ &\leq \frac{1}{1-\eta} \|C^1 - C^0\|. \end{aligned}$$

Letting  $t$  go to infinity on the left-hand side gives

$$\|C^* - C^0\| \leq \frac{1}{1-\eta} \|C^1 - C^0\|.$$

We next bound  $\|C^1 - C^0\|$ . Note that

$$C^1 = C^0 - \rho W((U^\top U) \odot (V^\top V) - I)$$

implying

$$\begin{aligned} \|C^1 - C^0\| &\leq \rho \|W\| (1 - \lambda_{\min}((U^\top U) \odot (V^\top V))) \\ &\leq \rho \left(1 + c\sqrt{\frac{r}{n}}\right) \frac{\kappa(\log n)\sqrt{r}}{n} \end{aligned} \tag{C.2}$$

where inequality (C.2) follows from Assumptions II and III.

Finally, plugging  $C^0 = \frac{1}{3}W$ , we get the desired result:

$$\begin{aligned} \left\|C^* - \frac{1}{3}W\right\| &\leq \frac{1 + c\sqrt{\frac{r}{n}}}{1 - \frac{\kappa(\log n)\sqrt{r}}{n}} \frac{\kappa(\log n)\sqrt{r}}{n} \\ &= \kappa(\log n) \left(1 + c\sqrt{\frac{r}{n}}\right) \left(1 + O\left(\kappa(\log n)\frac{\sqrt{r}}{n}\right)\right) \frac{\sqrt{r}}{n} \\ &\leq 2\kappa(\log n) \left(\frac{\sqrt{r}}{n} + c\frac{r}{n^{1.5}}\right) \end{aligned}$$

where the last line holds since  $1 + O(\kappa(\log n)\sqrt{r}/n) = 1 + o(1) \leq 2$  for sufficiently large  $n$ , when  $r \ll n^2/\kappa(\log n)^2$ . Similar arguments hold for  $\left\|A^* - \frac{1}{3}U\right\|$  and  $\left\|B^* - \frac{1}{3}V\right\|$ .  $\square$

## D Proof of Lemma 5.4

*Proof.* Noticing the symmetric role played by the three terms of the dual polynomial  $q(\mathbf{u}, \mathbf{v}, \mathbf{w})$  in (5.3), we focus on controlling one term,

$$\begin{aligned}
\sum_{p=1}^r \langle \boldsymbol{\alpha}_p^*, \mathbf{u} \rangle \langle \mathbf{v}_p^*, \mathbf{v} \rangle \langle \mathbf{w}_p^*, \mathbf{w} \rangle &\leq \max_p |\langle \boldsymbol{\alpha}_p^*, \mathbf{u} \rangle| \sum_p |\langle \mathbf{v}_p^*, \mathbf{v} \rangle| |\langle \mathbf{w}_p^*, \mathbf{w} \rangle| \\
&\leq \max_p |\langle \boldsymbol{\alpha}_p^*, \mathbf{u} \rangle| \sqrt{\sum_p |\langle \mathbf{v}_p^*, \mathbf{v} \rangle|^2} \sqrt{\sum_p |\langle \mathbf{w}_p^*, \mathbf{w} \rangle|^2} \\
&= \max_p |\langle \boldsymbol{\alpha}_p^*, \mathbf{u} \rangle| \|V^T \mathbf{v}\|_2 \|W^T \mathbf{w}\|_2 \\
&\leq \max_p |\langle \boldsymbol{\alpha}_p^*, \mathbf{u} \rangle| \|V\| \|W\|
\end{aligned}$$

Applying Lemma 5.3 to get

$$\begin{aligned}
\max_p |\langle \boldsymbol{\alpha}_p^*, \mathbf{u} \rangle| &\leq \max_p \left| \left\langle \boldsymbol{\alpha}_p^* - \frac{1}{3} \mathbf{u}_p^*, \mathbf{u} \right\rangle \right| + \frac{1}{3} \max_p |\langle \mathbf{u}_p^*, \mathbf{u} \rangle| \\
&\leq \max_p \left\| \boldsymbol{\alpha}_p^* - \frac{1}{3} \mathbf{u}_p^* \right\|_2 + \frac{1}{3} d \\
&= \frac{d}{3} + 2\kappa(\log n) \left( \frac{\sqrt{r}}{n} + c \frac{r}{n^{1.5}} \right). \tag{D.1}
\end{aligned}$$

Combining with Assumption II, we obtain

$$\sum_{p=1}^r \langle \boldsymbol{\alpha}_p^*, \mathbf{u} \rangle \langle \mathbf{v}_p^*, \mathbf{v} \rangle \langle \mathbf{w}_p^*, \mathbf{w} \rangle \leq \left[ \frac{d}{3} + 2\kappa(\log n) \left( \frac{\sqrt{r}}{n} + c \frac{r}{n^{1.5}} \right) \right] \left( 1 + c \sqrt{\frac{r}{n}} \right)^2.$$

Similar bounds hold for the other two terms. We therefore get an upper bound on the dual polynomial:

$$q(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq \left[ d + 6\kappa(\log n) \left( \frac{\sqrt{r}}{n} + c \frac{r}{n^{1.5}} \right) \right] \left( 1 + c \sqrt{\frac{r}{n}} \right)^2.$$

□

## E Proof of Lemma 5.5

*Proof.* By linearity of the dual polynomial (5.3) in each of its argument, we have

$$\begin{aligned}
q(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) &= q(\mathbf{u}_1^* \cos(\theta_1) + \mathbf{x} \sin(\theta_1), \mathbf{v}_1^* \cos(\theta_2) + \mathbf{y} \sin(\theta_2), \mathbf{w}_1^* \cos(\theta_3) + \mathbf{z} \sin(\theta_3)) \\
&= q(\mathbf{u}_1^*, \mathbf{v}_1^*, \mathbf{w}_1^*) \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) \\
&\quad + q(\mathbf{u}_1^*, \mathbf{v}_1^*, \mathbf{z}) \cos(\theta_1) \cos(\theta_2) \sin(\theta_3) \\
&\quad + q(\mathbf{u}_1^*, \mathbf{y}, \mathbf{w}_1^*) \cos(\theta_1) \sin(\theta_2) \cos(\theta_3) \\
&\quad + q(\mathbf{x}, \mathbf{v}_1^*, \mathbf{w}_1^*) \sin(\theta_1) \cos(\theta_2) \cos(\theta_3) \\
&\quad + q(\mathbf{u}_1^*, \mathbf{y}, \mathbf{z}) \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \\
&\quad + q(\mathbf{x}, \mathbf{v}_1^*, \mathbf{z}) \sin(\theta_1) \cos(\theta_2) \sin(\theta_3) \\
&\quad + q(\mathbf{x}, \mathbf{y}, \mathbf{w}_1^*) \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) \\
&\quad + q(\mathbf{x}, \mathbf{y}, \mathbf{z}) \sin(\theta_1) \sin(\theta_2) \sin(\theta_3). \tag{E.1}
\end{aligned}$$

Among these 8 terms, the first term is  $\cos(\theta_1)\cos(\theta_2)\cos(\theta_3)$  since  $q(\mathbf{u}_1^*, \mathbf{v}_1^*, \mathbf{w}_1^*) = 1$ . The next three terms are zero as, for example,

$$q(\mathbf{u}_1^*, \mathbf{v}_1^*, \mathbf{z}) = Q \bar{\times}_1 \mathbf{u}_1^* \bar{\times}_2 \mathbf{v}_1^* \bar{\times}_3 \mathbf{z} = \mathbf{w}_1^* \bar{\times}_3 \mathbf{z} = \mathbf{w}_1^{*\top} \mathbf{z} = 0,$$

where we have used  $Q \bar{\times}_1 \mathbf{u}_1^* \bar{\times}_2 \mathbf{v}_1^* = \mathbf{w}_1^*$ , the third line of (5.1).

The coefficients of the next three terms involving two sin functions are almost zero. To see this, we examine

$$\begin{aligned} q(\mathbf{x}, \mathbf{y}, \mathbf{w}_1^*) &= Q \bar{\times}_1 \mathbf{x} \bar{\times}_2 \mathbf{y} \bar{\times}_3 \mathbf{w}_1^* \\ &= \sum_{p=1}^r [\langle \alpha_p^*, \mathbf{x} \rangle \langle \mathbf{v}_p^*, \mathbf{y} \rangle \langle \mathbf{w}_p^*, \mathbf{w}_1^* \rangle + \langle \mathbf{u}_p^*, \mathbf{x} \rangle \langle \beta_p^*, \mathbf{y} \rangle \langle \mathbf{w}_p^*, \mathbf{w}_1^* \rangle + \langle \mathbf{u}_p^*, \mathbf{x} \rangle \langle \mathbf{v}_p^*, \mathbf{y} \rangle \langle \gamma_p^*, \mathbf{w}_1^* \rangle] \\ &= \mathbf{x}^\top [A^* \text{diag}(W^\top \mathbf{w}_1^*) V^\top + U \text{diag}(W^\top \mathbf{w}_1^*) B^{*\top} + U \text{diag}(C^{*\top} \mathbf{w}_1^*) V^\top] \mathbf{y} \\ &\approx \mathbf{x}^\top (\mathbf{u}_1^* \mathbf{v}_1^{*\top}) \mathbf{y} \\ &= 0. \end{aligned}$$

To get a precise estimate of  $q(\mathbf{x}, \mathbf{y}, \mathbf{w}_1^*)$ , we note

$$\begin{aligned} &\|A^* \text{diag}(W^\top \mathbf{w}_1^*) V^\top - \frac{1}{3} \mathbf{u}_1^* \mathbf{v}_1^{*\top}\| \\ &\leq \|A^* \text{diag}(W^\top \mathbf{w}_1^*) V^\top - \frac{1}{3} U \text{diag}(W^\top \mathbf{w}_1^*) V^\top\| + \|\frac{1}{3} U \text{diag}(W^\top \mathbf{w}_1^*) V^\top - \frac{1}{3} \mathbf{u}_1^* \mathbf{v}_1^{*\top}\| \end{aligned}$$

To bound the first term, we invoke Lemma 5.3 and Assumption I and II to get

$$\begin{aligned} \|A^* \text{diag}(W^\top \mathbf{w}_1^*) V^\top - \frac{1}{3} U \text{diag}(W^\top \mathbf{w}_1^*) V^\top\| &\leq \|A^* - \frac{1}{3} U\| \|\text{diag}(W^\top \mathbf{w}_1^*)\| \|V\| \\ &\leq 2\kappa(\log n) \left( \frac{\sqrt{r}}{n} + c \frac{r}{n^{1.5}} \right) \left( 1 + c \sqrt{\frac{r}{n}} \right) \\ &= 2\kappa(\log n) \frac{\sqrt{r}}{n} \left( 1 + c \sqrt{\frac{r}{n}} \right)^2 \end{aligned} \quad (\text{E.2})$$

for sufficiently large  $n$ . For the second term in the upper bound, we continue with

$$\begin{aligned} \frac{1}{3} \|U \text{diag}(W^\top \mathbf{w}_1^*) V^\top - \mathbf{u}_1^* \mathbf{v}_1^{*\top}\| &= \frac{1}{3} \|U \text{diag}(W^\top \mathbf{w}_1^* - \mathbf{e}_1) V^\top\| \\ &\leq \frac{1}{3} \max_p |\langle \mathbf{w}_p^*, \mathbf{w}_1^* \rangle - \delta_{p,1}| \|U\| \|V\| \\ &\leq \frac{\tau(\log n)}{3\sqrt{n}} \left( 1 + c \sqrt{\frac{r}{n}} \right)^2 \end{aligned} \quad (\text{E.3})$$

Combining (E.2) and (E.3) gives

$$\|A^* \text{diag}(W^\top \mathbf{w}_1^*) V^\top - \frac{1}{3} \mathbf{u}_1^* \mathbf{v}_1^{*\top}\| \leq \left[ 2\kappa(\log n) \frac{\sqrt{r}}{n} + \frac{\tau(\log n)}{3\sqrt{n}} \right] \left( 1 + c \sqrt{\frac{r}{n}} \right)^2$$

Similar bound holds for  $\|U \text{diag}(W^\top \mathbf{w}_1^*) B^{*\top} - \frac{1}{3} \mathbf{u}_1^* \mathbf{v}_1^{*\top}\|$  and  $\|U \text{diag}(C^{*\top} \mathbf{w}_1^*) V^\top - \frac{1}{3} \mathbf{u}_1^* \mathbf{v}_1^{*\top}\|$ . Finally, we obtain

$$\begin{aligned} q(\mathbf{x}, \mathbf{y}, \mathbf{w}_1^*) &= \mathbf{x}^\top (A^* \text{diag}(W^\top \mathbf{w}_1^*) V^\top + U \text{diag}(W^\top \mathbf{w}_1^*) B^{*\top} + U \text{diag}(C^{*\top} \mathbf{w}_1^*) V^\top - \mathbf{u}_1^* \mathbf{v}_1^{*\top} + \mathbf{u}_1^* \mathbf{v}_1^{*\top}) \mathbf{y} \\ &\leq \|A^* \text{diag}(W^\top \mathbf{w}_1^*) V^\top + U \text{diag}(W^\top \mathbf{w}_1^*) B^{*\top} + U \text{diag}(C^{*\top} \mathbf{w}_1^*) V^\top\| \\ &\leq \left[ 6\kappa(\log n) \frac{\sqrt{r}}{n} + \frac{\tau(\log n)}{\sqrt{n}} \right] \left( 1 + c \sqrt{\frac{r}{n}} \right)^2. \end{aligned}$$

The same bound holds for  $q(\mathbf{x}, \mathbf{v}_1^*, \mathbf{z})$  and  $q(\mathbf{u}_1^*, \mathbf{y}, \mathbf{z})$ .

The coefficient of the last term of (E.1),  $q(\mathbf{x}, \mathbf{y}, \mathbf{z})$ , is bounded by the tensor spectral norm of  $Q$ , and should be close to constant as  $Q$  is close to  $\sum_{p=1}^r \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^*$ , the spectral norm of which is  $1 + o(1)$  (see Lemma E.2). Let us estimate the tensor spectral norm  $\|Q\|$  precisely. First of all, bound the difference between  $Q$  and  $\sum_{p=1}^r \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^*$ :

$$\begin{aligned} \left\| Q - \sum_{p=1}^r \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* \right\| &\leq \left\| \sum_{p=1}^r (\alpha_p^* - \frac{1}{3} \mathbf{u}_p^*) \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* \right\| \\ &\quad + \left\| \sum_{p=1}^r \mathbf{u}_p^* \otimes (\beta_p^* - \frac{1}{3} \mathbf{v}_p^*) \otimes \mathbf{w}_p^* \right\| \\ &\quad + \left\| \sum_{p=1}^r \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes (\gamma_p^* - \frac{1}{3} \mathbf{w}_p^*) \right\| \\ &= \Pi_1 + \Pi_2 + \Pi_3. \end{aligned}$$

Consider

$$\begin{aligned} \Pi_1 &= \sup_{\|\mathbf{y}\|=\|\mathbf{z}\|=1} \left\| (A^* - \frac{1}{3}U) [\langle \mathbf{v}_p^*, \mathbf{y} \rangle \langle \mathbf{w}_p^*, \mathbf{z} \rangle]_{1 \leq p \leq n} \right\| \\ &\leq \left\| A^* - \frac{1}{3}U \right\| \left\| [\langle \mathbf{v}_p^*, \mathbf{y} \rangle \langle \mathbf{w}_p^*, \mathbf{z} \rangle]_{1 \leq p \leq n} \right\|_2 \\ &\leq 2\kappa(\log n) \left( \frac{\sqrt{r}}{n} + c \frac{r}{n^{1.5}} \right) \left\| [\langle \mathbf{v}_p^*, \mathbf{y} \rangle \langle \mathbf{w}_p^*, \mathbf{z} \rangle]_{1 \leq p \leq n} \right\|_2, \end{aligned} \tag{E.4}$$

where (E.4) follows from Lemma 5.3.

Note that

$$\begin{aligned} \left\| [\langle \mathbf{v}_p^*, \mathbf{y} \rangle \langle \mathbf{w}_p^*, \mathbf{z} \rangle]_{1 \leq p \leq n} \right\|_2 &= \left( \sum_p |\langle \mathbf{v}_p^*, \mathbf{y} \rangle|^2 |\langle \mathbf{w}_p^*, \mathbf{z} \rangle|^2 \right)^{1/2} \\ &\leq \left( \sum_p |\langle \mathbf{v}_p^*, \mathbf{y} \rangle|^4 \right)^{1/4} \left( \sum_p |\langle \mathbf{w}_p^*, \mathbf{z} \rangle|^4 \right)^{1/4} \\ &\leq \|V^\top\|_{2 \rightarrow 4} \|W^\top\|_{2 \rightarrow 4} \end{aligned}$$

where  $\|V^\top\|_{2 \rightarrow 4}, \|W^\top\|_{2 \rightarrow 4}$  are bounded according to Lemma E.1.

**Lemma E.1.** *Under Assumptions I and II,*

$$\begin{aligned} \|U^\top\|_{2 \rightarrow 3} &\leq 1 + \frac{1}{3} \tau(\log n) n^{-r_c}, \\ \|U^\top\|_{2 \rightarrow 4} &\leq 1 + \frac{1}{3} \tau(\log n) n^{-r_c}, \end{aligned}$$

if  $r \leq n^{1.25-1.5r_c}$  with  $r_c \in (0, 1/6)$ . The same bounds hold for  $V$  and  $W$ .

As a consequence of Lemma E.1, when  $r \leq n^{1.25-1.5r_c}$  with  $r_c \in (0, 1/6)$ ,

$$\Pi_1 \leq 2\kappa(\log n) \left( \frac{\sqrt{r}}{n} + c \frac{r}{n^{1.5}} \right) (1 + o(1)) \leq 8\kappa(\log n) \max \left\{ \frac{\sqrt{r}}{n}, c \frac{r}{n^{1.5}} \right\}.$$

The same bound also holds for  $\Pi_2$  and  $\Pi_3$ . Therefore, we have

$$\|Q\| \leq \left\| \sum_{p=1}^r \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* \right\| + 24\kappa(\log n) \max \left\{ \frac{\sqrt{r}}{n}, c \frac{r}{n^{1.5}} \right\}.$$

**Lemma E.2.** *Under Assumptions I and II, if  $r \leq n^{1.25-1.5r_c}$  with  $r_c \in (0, 1/6)$ ,*

$$\left\| \sum_{p=1}^r \mathbf{u}_p^* \otimes \mathbf{v}_p^* \otimes \mathbf{w}_p^* \right\| \leq 1 + 2\tau(\log n)n^{-r_c}.$$

We hence obtain an upper bound on  $q(\mathbf{x}, \mathbf{y}, \mathbf{z})$

$$q(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq \|Q\| \leq 1 + 2\tau(\log n)n^{-r_c} + 24\kappa(\log n) \max\{\sqrt{r}n^{-1}, crn^{-1.5}\}.$$

The requirement  $r \leq n^{1.25-1.5r_c}$  and  $r_c \in (0, \frac{1}{6})$  imply that

$$\begin{aligned} \left[ 6\kappa(\log n) \frac{\sqrt{r}}{n} + \frac{\tau(\log n)}{\sqrt{n}} \right] \left( 1 + c \sqrt{\frac{r}{n}} \right)^2 &\leq O \left( \max\{\kappa(\log n)n^{-1/8-9/4r_c}, \tau(\log n)n^{-1/4-3/2r_c}\} \right) \ll n^{-r_c} \\ \kappa(\log n)rn^{-1.5} &\leq \kappa(\log n)n^{-1/4-3/2r_c} \ll n^{-r_c} \\ \kappa(\log n)\sqrt{r}n^{-1} &\leq \kappa(\log n)n^{-3/8-3/4r_c} \ll n^{-r_c}. \end{aligned}$$

Summarizing these arguments, we obtain the following estimates

$$\begin{aligned} \max\{q(\mathbf{u}_1^*, \mathbf{y}, \mathbf{z}), q(\mathbf{x}, \mathbf{v}_1^*, \mathbf{z}), q(\mathbf{x}, \mathbf{y}, \mathbf{w}_1^*)\} &= o(n^{-r_c}) \\ q(\mathbf{x}, \mathbf{y}, \mathbf{z}) &\leq 1 + 2\tau(\log n)n^{-r_c} + o(n^{-r_c}). \end{aligned}$$

Plugging these into the expression (5.8) leads to

$$q(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) \leq \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) + \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) + 2\tau(\log n)n^{-r_c} + o(n^{-r_c}).$$

Enlarging the dominating term gives the desired upper bound:

$$q(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) \leq \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) + \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) + 4\tau(\log n)n^{-r_c}.$$

□

## E.1 Proof of Lemma E.1

*Proof.* The proof refines the one for Lemma 4 of [17]. We only prove it for  $U$  since the same argument applies to  $W$  and  $V$ . We start with a general  $p > 2$  and assume  $\|U^\top\|_{2 \rightarrow p} > 1$ ; otherwise, we are done. Thus,

$$\|U^\top\|_{2 \rightarrow p} \leq \|U^\top\|_{2 \rightarrow p}^p = \sup_{\mathbf{x} \in \mathbb{S}^{n-1}} \|U^\top \mathbf{x}\|_p^p = \|U^\top \mathbf{x}^*\|_p^p. \quad (\text{E.5})$$

where  $\mathbf{x}^* \in \mathbb{S}^{n-1}$  is the optimal solution of  $\sup_{\mathbf{x} \in \mathbb{S}^{n-1}} \|U^\top \mathbf{x}\|_p^p$ . Let  $S$  be the indices for the largest (in absolute value)  $L$  entries of  $U^\top \mathbf{x}^*$  and note that

$$\|U^\top \mathbf{x}^*\|_p^p = \|U_S^\top \mathbf{x}^*\|_p^p + \|U_{S^c}^\top \mathbf{x}^*\|_p^p. \quad (\text{E.6})$$

(i) Bound the first term:

$$\begin{aligned} \|U_S^\top \mathbf{x}^*\|_p^p &\leq \|U_S^\top \mathbf{x}^*\|_2^2 \\ &\leq \|U_S U_S^\top\| \\ &\leq 1 + \sum_{i \neq j} |\langle \mathbf{u}_i, \mathbf{u}_j \rangle| \end{aligned} \quad (\text{E.7})$$

$$\leq 1 + \frac{L\tau(\log n)}{\sqrt{n}} \quad (\text{E.8})$$

where the first inequality is due to  $|\mathbf{u}_i^\top \mathbf{x}^\star| \leq 1$  for any  $i$ , the inequality (E.7) follows from *Gershgorin's circle theorem* and the line (E.8) holds by Assumption I. Note that  $\frac{L\kappa(\log n)}{\sqrt{n}} = o(1)$  when  $L \ll n^{0.5}/\tau(\log n)$  and the resulting upper-bound of  $\|U_S^\top \mathbf{x}^\star\|_p^p$  is independent of  $p$ .

(ii) Bound the second term: first note that

$$\begin{aligned} \min_{i \in S} |\mathbf{u}_i^\top \mathbf{x}^\star|^2 &\leq \frac{1}{L} \sum_{i \in S} |\mathbf{u}_i^\top \mathbf{x}^\star|^2 \\ &\leq \frac{1}{L} \|U_S U_S^\top\| \|\mathbf{x}^\star\|^2 \\ &\leq \frac{1}{L} (1 + o(1)) \end{aligned}$$

and  $o(1) \leq 1$  for sufficiently large  $n$ . We conclude that

$$\max_{i \in S^c} |\mathbf{u}_i^\top \mathbf{x}^\star|^2 \leq \min_{i \in S} |\mathbf{u}_i^\top \mathbf{x}^\star|^2 \leq \frac{2}{L}$$

since  $S$  consists the indices of the  $L$  largest (in absolute value) elements of  $U^\top \mathbf{x}^\star$ . As a consequence, we have

$$\begin{aligned} \|U_{S^c}^\top \mathbf{x}^\star\|_p^p &= \sum_{i \notin S} |\mathbf{u}_i^\top \mathbf{x}^\star|^p \\ &\leq \left( \max_{i \notin S} |\mathbf{u}_i^\top \mathbf{x}^\star|^{p-2} \right) \sum_{i \notin S} |\mathbf{u}_i^\top \mathbf{x}^\star|^2 \\ &= \left( \max_{i \notin S} |\mathbf{u}_i^\top \mathbf{x}^\star|^{p-2} \right) \|U_{S^c}^\top \mathbf{x}^\star\|_2^2 \\ &\leq \left( \frac{2}{L} \right)^{\frac{p}{2}-1} \left( 1 + c\sqrt{\frac{r}{n}} \right)^2 \end{aligned} \tag{E.9}$$

where the inequality (E.9) holds by the fact that  $\|U_{S^c}^\top \mathbf{x}^\star\|_2^2 \leq \|U\|^2$  and Assumption II. In particular, when  $p = 3$ , we have

$$\begin{aligned} \|U_{S^c}^\top \mathbf{x}^\star\|_3^3 &\leq \sqrt{2} L^{-0.5} \left( 1 + c\sqrt{\frac{r}{n}} \right)^2 \\ &\leq 4\sqrt{2} L^{-0.5} \max \left\{ 1, c^2 \frac{r}{n} \right\}. \end{aligned} \tag{E.10}$$

Combining (E.8) and (E.10) yields

$$\|U^\top \mathbf{x}^\star\|_3^3 \leq 1 + \frac{L\tau(\log n)}{\sqrt{n}} + 4\sqrt{2} L^{-0.5} \max \left\{ 1, c^2 \frac{r}{n} \right\}.$$

By choosing  $L = \frac{1}{2} n^{0.5-r_c}$  (which satisfies  $L \ll n^{0.5}/\tau(\log n)$ ) and  $r \leq n^{1.25-1.5r_c}$  with  $r_c \in (0, 1/6)$ , we have

$$\begin{aligned} \|U^\top \mathbf{x}^\star\|_3^3 &\leq 1 + \frac{1}{2} \tau(\log n) n^{-r_c} + 8 \max \{ n^{0.5r_c-0.25}, c^2 n^{-r_c} \} \\ &= 1 + \left( \frac{1}{2} \tau(\log n) + 8 \right) n^{-r_c} \\ &\leq 1 + \tau(\log n) n^{-r_c} \end{aligned}$$

where the last inequality holds since  $16 \leq \tau(\log n)$  for sufficient large  $n$ .

Finally, by (E.5), we obtain

$$\|U^\top\|_{2 \rightarrow 3}^3 \leq 1 + \tau(\log n) n^{-r_c}, \text{ if } r \leq n^{1.25-1.5r_c} \text{ with } r_c \in (0, 1/6).$$



Denote  $t = \tau(\log n)n^{-r_c} > 0$ . Direct computation gives that  $(1+t)^{1/3} \leq 1 + \frac{t}{3}$  for any  $t > 0$ , implying that

$$\|U^\top\|_{2 \rightarrow 3} \leq 1 + \frac{1}{3}\tau(\log n)n^{-r_c}, \text{ if } r \leq n^{1.25-1.5r_c} \text{ with } r_c \in (0, 1/6).$$

Applying similar arguments to the case  $p = 4$ , we have

$$\|U^\top\|_{2 \rightarrow 4} \leq 1 + \frac{1}{3}\tau(\log n)n^{-r_c}, \text{ if } r \leq n^{1.5-2r_c} \text{ with } r_c \in (0, 1/4).$$

Since  $1.25 - 1.5r_c < 1.5 - 2r_c$  for  $r_c \in (0, \frac{1}{4})$ , we merge the conditions on  $r$  and  $r_c$  to get the desired result.  $\square$

## E.2 Proof of Lemma E.2

*Proof.* The proof modifies the one for Lemma 5 of [17]. For any vectors  $\mathbf{f}, \mathbf{g}, \mathbf{h}$  of the same dimension, we have

$$\begin{aligned} \sum_i \mathbf{f}(i)\mathbf{g}(i)\mathbf{h}(i) &\leq \sum_i |\mathbf{f}(i)\mathbf{g}(i)\mathbf{h}(i)| \\ &\leq \left( \sum_i |\mathbf{f}(i)|^3 \right)^{1/3} \left( \sum_i |\mathbf{g}(i)\mathbf{h}(i)|^{3/2} \right)^{2/3} \end{aligned} \quad (\text{E.11})$$

$$\leq \|\mathbf{f}\|_3 \|\mathbf{g}\|_3 \|\mathbf{h}\|_3 \quad (\text{E.12})$$

where both the inequities (E.11) and (E.12) are due to Hölder's inequality for  $p = 3, q = 3/2$  and  $p = q = 2$ , respectively.

Note that

$$\begin{aligned} \left\| \sum_{p=1}^r \mathbf{u}_p^\star \otimes \mathbf{v}_p^\star \otimes \mathbf{w}_p^\star \right\| &= \sup_{\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{K}} \sum_i (U^\top \mathbf{a})_i (V^\top \mathbf{b})_i (W^\top \mathbf{c})_i \\ &\leq \sup_{\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{K}} \|U^\top \mathbf{a}\|_3 \|V^\top \mathbf{b}\|_3 \|W^\top \mathbf{c}\|_3 \end{aligned} \quad (\text{E.13})$$

$$\begin{aligned} &\leq \|U^\top\|_{2 \rightarrow 3} \|V^\top\|_{2 \rightarrow 3} \|W^\top\|_{2 \rightarrow 3} \\ &\leq 1 + \tau(\log n)n^{-r_c} + O(\tau(\log n)^2 n^{-2r_c}). \\ &\leq 1 + 2\tau(\log n)n^{-r_c} \end{aligned} \quad (\text{E.14})$$

The inequality (E.13) follows from (E.12) and the inequality (E.14) arises from Lemma E.1 when  $r \leq n^{1.25-1.5r_c}$  with  $r_c \in (0, \frac{1}{6})$ .  $\square$

## F Proof of Lemma 5.6

Lemma 5.6 relies on the following lemma.

**Lemma F.1** (Nearest Region Guarantee). *Under Assumptions I, II, III and if  $r \leq n^{1.25-1.5r_c}$  with  $r_c \in (0, \frac{1}{6})$ , we have*

$$q(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) \leq 1, \forall \theta_i \in \left[0, \frac{\sqrt{2}-1}{3}\right]$$

with equality only if  $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$ .

Now, we are ready to show Lemma 5.6.

*Proof.* By Lemma 5.5, if  $r \leq n^{1.25-1.5r_c}$  with  $r_c \in (0, \frac{1}{6})$  and under Assumptions I, II, III,

$$\begin{aligned} q(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) &\leq \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) + \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) + 4\tau(\log n)n^{-r_c} \\ &\leq \frac{1}{3}(\cos^3(\theta_1) + \cos^3(\theta_2) + \cos^3(\theta_3)) + \frac{1}{3}(\sin^3(\theta_1) + \sin^3(\theta_2) + \sin^3(\theta_3)) + 4\tau(\log n)n^{-r_c} \\ &= \frac{1}{3}(\cos^3(\theta_1) + \sin^3(\theta_1) + 4\tau(\log n)n^{-r_c}) + \frac{1}{3}(\cos^3(\theta_2) + \sin^3(\theta_2) + 4\tau(\log n)n^{-r_c}) \\ &\quad + \frac{1}{3}(\cos^3(\theta_3) + \sin^3(\theta_3) + 4\tau(\log n)n^{-r_c}) \end{aligned}$$

with the second line following from the GM-AM inequality. Therefore, to ensure  $q(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) < 1$ , one needs

$$\cos^3(\theta_i) + \sin^3(\theta_i) < 1 - 4\tau(\log n)n^{-r_c}, \text{ for } i = 1, 2, 3. \quad (\text{F.1})$$

Define  $f(x) := \cos^3(x) + \sin^3(x)$ , whose plot over  $[0, \pi]$  is shown in Figure 5. We observe that when  $b \in (\frac{1}{\sqrt{2}}, 1)$

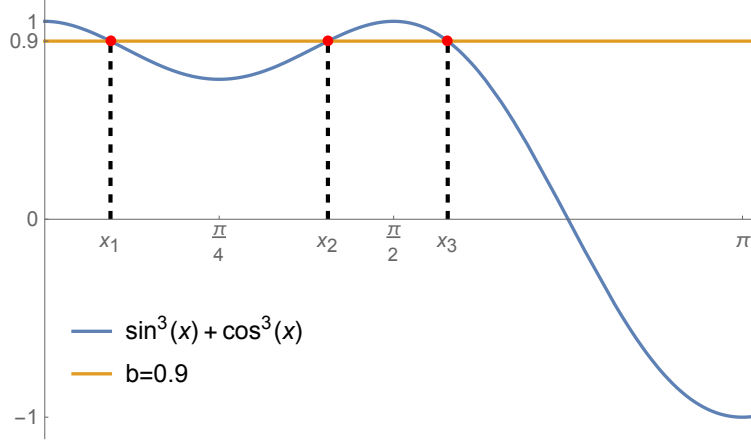


Figure 5: Property of the function  $f(x) = \sin(x)^3 + \cos(x)^3$ , for  $x \in [0, \pi]$ .

( $\frac{1}{\sqrt{2}}$  is the minimum of  $f(x)$  over  $[0, \pi/2]$ ), there are three roots located in  $(0, \pi)$  of the equation  $f(x) = b$ , denoted by  $x_1, x_2, x_3$ . For sufficiently large  $n$ ,  $1 - 4\tau(\log n)n^{-r_c} \in (\frac{1}{\sqrt{2}}, 1)$ . The feasible set of the inequality  $f(x) < b$  can be expressed as

$$(x_1, x_2) \cup (x_3, \pi). \quad (\text{F.2})$$

It is straightforward to show and can be observed from Figure 5 that

$$\begin{aligned} \frac{\pi}{4} - x_1 &= x_2 - \frac{\pi}{4} \\ \frac{\pi}{2} - x_2 &\geq x_3 - \frac{\pi}{2} \end{aligned}$$

implying a subset of the feasible set (F.2) is given by

$$(x_1, \frac{\pi}{2} - x_1) \cup (\frac{\pi}{2} + x_1, \pi).$$

Next, we will calculate an upper-bound on  $x_1$ , denoted by  $\delta$ . As a consequence, the set

$$(\delta, \frac{\pi}{2} - \delta) \cup (\frac{\pi}{2} + \delta, \pi)$$

becomes a subset of the feasible set (F.2). Observing that  $x_1 \in (0, \frac{\pi}{4})$  and recognizing

$$\sin^3(x) + \cos^3(x) \leq 1 - 0.15x^2, \text{ when } x \in (0, \frac{\pi}{4}), \quad (\text{F.3})$$

we thus use the solution of

$$1 - 0.15x^2 = 1 - 4\tau(\log n)n^{-r_c}$$

located in  $(0, \frac{\pi}{4})$  as an upper-bound of  $x_1$ , which is  $\delta = \sqrt{\frac{80}{3}\tau(\log n)n^{-r_c/2}}$ . With such a choice of  $\delta$ , we have  $q(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) < 1$  when

$$\theta_i \in [\delta, \frac{\pi}{2} - \delta] \cup [\frac{\pi}{2} + \delta, \pi], \quad i = 1, 2, 3. \quad (\text{F.4})$$

Since  $\delta = \sqrt{\frac{80}{3}\kappa(\log n)n^{-0.5r_c}}$ , it is obvious that  $\delta \leq \frac{\sqrt{2}-1}{3}$  for large  $n$ . We therefore can combine (F.4) and Lemma F.1 to get that when

$$\theta_i \in [0, \frac{\pi}{2} - \delta] \cup [\frac{\pi}{2} + \delta, \pi], \quad i = 1, 2, 3, \quad (\text{F.5})$$

the function  $q(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) \leq 1$  with equality only if  $(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u}_1^*, \mathbf{v}_1^*, \mathbf{w}_1^*)$ . Note that the condition (F.5) is equivalent to

$$|\cos(\theta_i)| \geq \sin(\delta), \quad i = 1, 2, 3. \quad (\text{F.6})$$

We use a sufficient condition of (F.6),

$$|\cos(\theta_i)| \geq \delta, \quad i = 1, 2, 3. \quad (\text{F.7})$$

since and  $\delta \geq \sin(\delta)$  for  $\delta \in [0, \pi]$ . Recognizing that  $\cos(\theta_1) = \langle \mathbf{u}_1^*, \mathbf{u}(\theta_1) \rangle$ , etc., the condition (F.7) becomes

$$|\langle \mathbf{u}_1^*, \mathbf{u} \rangle| \geq \delta, \quad |\langle \mathbf{v}_1^*, \mathbf{v} \rangle| \geq \delta, \quad |\langle \mathbf{w}_1^*, \mathbf{w} \rangle| \geq \delta,$$

which precisely describes the near-region  $\mathcal{N}_1(\delta; \mathbf{x}, \mathbf{y}, \mathbf{z})$ . Due to the arbitrariness of  $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ , we arrive at the desired result for the lemma for  $\mathcal{N}_1(\delta)$ .  $\square$

## F.1 Proof of Lemma F.1

*Proof.* Denote

$$\begin{aligned} F(\theta_1, \theta_2, \theta_3) &:= q(\mathbf{u}(\theta_1), \mathbf{v}(\theta_2), \mathbf{w}(\theta_3)) \\ &= \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) \\ &\quad + q(\mathbf{u}_1^*, \mathbf{y}, \mathbf{z}) \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \\ &\quad + q(\mathbf{x}, \mathbf{y}, \mathbf{w}_1^*) \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) \\ &\quad + q(\mathbf{x}, \mathbf{v}_1^*, \mathbf{z}) \sin(\theta_1) \cos(\theta_2) \sin(\theta_3) \\ &\quad + q(\mathbf{x}, \mathbf{y}, \mathbf{z}) \sin(\theta_1) \sin(\theta_2) \sin(\theta_3). \end{aligned}$$

Let us apply the first order Taylor expansion to  $F(\theta_1, \theta_2, \theta_3)$  over the region  $[0, \theta_0] \times [0, \theta_0] \times [0, \theta_0]$  with  $\theta_0 \in (0, \pi/2)$  to be determined later,

$$\begin{aligned} F(\theta_1, \theta_2, \theta_3) &= F(0, 0, 0) + \boldsymbol{\theta}^T \nabla F(\xi_1, \xi_2, \xi_3) \\ &\geq 1 - \|\boldsymbol{\theta}\|_1 \sup_{|\xi_1|, |\xi_2|, |\xi_3| \leq \theta_0} \|\nabla F(\xi_1, \xi_2, \xi_3)\|_\infty \end{aligned}$$

where  $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$ . Since

$$\begin{aligned} \frac{\partial}{\partial \theta_1} F(\xi_1, \xi_2, \xi_3) &= -\sin(\xi_1) \cos(\xi_2) \cos(\xi_3) \\ &\quad - q(\mathbf{u}_1^*, \mathbf{y}, \mathbf{z}) \sin(\xi_1) \sin(\xi_2) \sin(\xi_3) \\ &\quad + q(\mathbf{x}, \mathbf{v}_1^*, \mathbf{z}) \cos(\xi_1) \cos(\xi_2) \sin(\xi_3) \\ &\quad + q(\mathbf{x}, \mathbf{y}, \mathbf{w}_1^*) \cos(\xi_1) \sin(\xi_2) \cos(\xi_3) \\ &\quad + q(\mathbf{x}, \mathbf{y}, \mathbf{z}) \cos(\xi_1) \sin(\xi_2) \sin(\xi_3), \end{aligned}$$

we apply the estimates (5.9) and (5.10) in Lemma 5.5 to get

$$\begin{aligned} \left| \frac{\partial}{\partial \theta_1} F(\xi_1, \xi_2, \xi_3) \right| &\leq |\sin(\theta_0)| + o(n^{-r_c})(|\sin(\theta_0)|^3 + 2|\sin(\theta_0)|) + (1 + O(n^{-r_c}))|\sin(\theta_0)|^2 \\ &\leq |\sin(\theta_0)| + |\sin(\theta_0)|^2 + o(1) \\ &\leq 3|\sin(\theta_0)|, \end{aligned} \tag{F.8}$$

where the inequality (F.8) follows from the facts that  $|\sin(\theta_0)|^2 \leq |\sin(\theta_0)|$  and  $o(1) \leq |\sin(\theta_0)|$  for sufficiently large  $n$ . The same bound holds for  $\left| \frac{\partial}{\partial \theta_2} F(\xi_1, \xi_2, \xi_3) \right|$  and  $\left| \frac{\partial}{\partial \theta_3} F(\xi_1, \xi_2, \xi_3) \right|$ . We therefore have

$$F(\theta_1, \theta_2, \theta_3) \geq 1 - 3\|\boldsymbol{\theta}\|_1 |\sin(\theta_0)| \geq 1 - 9\theta_0^2. \tag{F.9}$$

Let us compute the second order Taylor expansion of  $F(\theta_1, \theta_2, \theta_3)$ :

$$F(\theta_1, \theta_2, \theta_3) = F(0, 0, 0) + \boldsymbol{\theta}^T \nabla F(0, 0, 0) + \frac{1}{2} \boldsymbol{\theta}^T \nabla^2 F(\xi_1, \xi_2, \xi_3) \boldsymbol{\theta}$$

where  $(\xi_1, \xi_2, \xi_3) \in [0, \theta_0] \times [0, \theta_0] \times [0, \theta_0]$ . As a consequence of the dual polynomial construction process, we have  $F(0, 0, 0) = 1$  and  $\nabla F(0, 0, 0) = 0$ , implying

$$F(\theta_1, \theta_2, \theta_3) = 1 + \frac{1}{2} \boldsymbol{\theta}^T \nabla^2 F(\xi_1, \xi_2, \xi_3) \boldsymbol{\theta}.$$

Therefore, as long as we can find  $\theta_0$  such that the Hessian matrix  $\nabla^2 F$  is negative definite over the region  $[0, \theta_0] \times [0, \theta_0] \times [0, \theta_0]$ , then  $F(\theta_1, \theta_2, \theta_3) \leq 1$  for any  $(\theta_1, \theta_2, \theta_3) \in [0, \theta_0] \times [0, \theta_0] \times [0, \theta_0]$  with equality only if  $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$ .

We next estimate the Hessian matrix  $\nabla^2 F(\xi_1, \xi_2, \xi_3)$ . Direct computation gives

$$\nabla^2 F(\xi_1, \xi_2, \xi_3) = \begin{bmatrix} -F(\xi_1, \xi_2, \xi_3) & * & * \\ * & -F(\xi_1, \xi_2, \xi_3) & * \\ * & * & -F(\xi_1, \xi_2, \xi_3) \end{bmatrix}$$

whose off-diagonal elements are non-symmetric partial derivatives of  $F$ , for example,

$$\begin{aligned} \frac{\partial^2}{\partial \theta_1 \partial \theta_2} F(\theta_1, \theta_2, \theta_3) &= \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) \\ &\quad - q(\mathbf{u}_1^*, \mathbf{y}, \mathbf{z}) \sin(\theta_1) \cos(\theta_2) \sin(\theta_3) \\ &\quad + q(\mathbf{x}, \mathbf{y}, \mathbf{w}_1^*) \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) \\ &\quad - q(\mathbf{x}, \mathbf{v}_1^*, \mathbf{z}) \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \\ &\quad + q(\mathbf{x}, \mathbf{y}, \mathbf{z}) \cos(\theta_1) \cos(\theta_2) \sin(\theta_3), \end{aligned}$$

which implies

$$\begin{aligned} \left| \frac{\partial^2}{\partial \theta_1 \partial \theta_2} F(\theta_1, \theta_2, \theta_3) \right| &\leq |\sin(\theta_0)|^2 + o(n^{-r_c})(1 + 2|\sin(\theta_0)|^2) + (1 + O(n^{-r_c}))|\sin(\theta_0)| \\ &\leq |\sin(\theta_0)| + |\sin(\theta_0)|^2 + o(1) \\ &\leq 3|\sin(\theta_0)|. \end{aligned}$$

The same bound holds for other mixed partial derivatives  $\left| \frac{\partial^2}{\partial \theta_i \partial \theta_j} F(\theta_1, \theta_2, \theta_3) \right|$  with  $i, j = 1, 2, 3$  and  $i \neq j$ .  
To make  $\nabla^2 F(\xi_1, \xi_2, \xi_3)$  negative definite, by Gershgorin's circle theorem and the bound (F.9), one only needs

$$-F(\xi_1, \xi_2, \xi_3) + 6|\sin(\theta_0)| \leq -1 + 9\theta_0^2 + 6\theta_0 < 0$$

which is satisfied for  $\theta_0 \in [0, \frac{\sqrt{2}-1}{3})$ . This completes the proof.  $\square$

## F.2 Show (F.3)

*Proof.* First, since  $\sin(x) \leq x$  for  $x \in (0, \frac{\pi}{4})$ , we have

$$\sin^3(x) \leq x^3 \text{ for } x \in (0, \frac{\pi}{4}). \quad (\text{F.10})$$

Second,

$$\cos^3(x) = 1 - \frac{3}{2}x^2 + \frac{7}{8}x^4 + O(x^5).$$

Combining it with the alternating sign property generates

$$\cos^3(x) \leq 1 - \frac{3}{2}x^2 + \frac{7}{8}x^4. \quad (\text{F.11})$$

Combining the bounds (F.10) and (F.11) yields

$$\cos^3(x) + \sin^3(x) \leq 1 - \frac{3}{2}x^2 + x^3 + \frac{7}{8}x^4 \quad (\text{F.12})$$

Therefore, to show (F.3), it suffices to show

$$1 - \frac{3}{2}x^2 + x^3 + \frac{7}{8}x^4 \leq 1 - 0.15x^2 \quad (\text{F.13})$$

holds for  $x \in (0, \frac{\pi}{4})$ . This is true since the solution set of (F.13) is  $\left[0, \frac{2(\sqrt{2290}-20)}{35}\right]$  which contains  $(0, \frac{\pi}{4})$ .  $\square$